Class 1: Stationary Time Series Analysis

Macroeconometrics - Fall 2009

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December 15, 2009

Outline

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1. Covariance-Stationary Processes
2. Wold Decomposition Theorem
3. ARMA Models
4. AR(1) Model

Stochastic Process

- Stochastic process: a collection of random variables

$$\{\ldots, Y_{-1}, Y_0, Y_1, Y_2, \ldots, Y_T, \ldots\} = \{Y_t\}_{-\infty}^{\infty}.$$

- Observed series $\{y_1, y_2, \ldots, y_T\}$ – realizations of a stochastic process.

  We want a model for $\{Y_t\}_{\infty}^{-\infty}$ to explain observed realizations $\{y_t\}_1^T$.

Covariance-Stationary

Definition

$\{Y_t\}$ is covariance-stationary (weak stationary) if

(i) $E[Y_t] = \mu \quad \forall t$

(ii) $Cov(Y_t, Y_{t-j}) = E[(Y_t - \mu)(Y_{t-j} - \mu)] = \gamma_j \quad ,\forall t, j$

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<u>Note:</u> mean is time-invariant

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<u>Note:</u> covariance doesn't depend on t

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Note: $Var(Y_t) = \gamma_0$ – variance also constant.

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- Normality $\Rightarrow$ strong stationarity: whole distribution depends on the first two moments.

Examples

1. $Y_t \sim iid(0, \sigma^2) \Rightarrow \{Y_t\}$ white noise (WN).
2. $Y_t \sim iidN(0, \sigma^2) \Rightarrow$ Gaussian white noise.

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$

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t - time dummy

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deterministic part

Nonstationary Processes

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stochastic component

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$
   - $E[Y_t] = \beta \cdot t$ depends on t
   - But, $X_t = Y_t - \beta t$ is covariance stationary.

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1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$
   - $E[Y_t] = \beta \cdot t$ depends on t
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2. $Y_t = Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN$, Y_0 constant
   - Random walk

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$
   - $E[Y_t] = \beta \cdot t$ depends on t
   - But, $X_t = Y_t - \beta t$ is covariance stationary.

2. $Y_t = Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN$, Y_0 constant
   - Solving recursively :

$$Y_t = \sum_{j=1}^{t} \varepsilon_j + Y_0.$$

   - $E[Y_t] = Y_0$, time-invariant mean.
   - But $Var(Y_t) = t \cdot \sigma^2$ depends on t.
   - $X_t = Y_t - Y_{t-1}$ is covariance stationary.

Wold's Decomposition Theorem

Any covariance stationary $\{Y_t\}$ has infinite order, moving-average representation:

$$Y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \kappa_t, \qquad \psi_0 = 1, \varepsilon_t \sim WN.$$

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- Linear combination of ε_s (innovations over time)
- Weights does not depend on time t, they only depend on j, i.e. how long ago the shock ε occurred.

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- $\sum_{j=0}^{\infty} \psi_j^2 < \infty$,
- $\varepsilon_t \sim WN(O, \sigma^2)$,
- κ_t deterministic term (perfectly forecastable).
- Example: $\kappa_t = \mu$, constant mean.

Wold's Decomposition Theorem - Illustration

Let $X_t = Y_t - \kappa_t$. Then,

$$
\begin{aligned}
E[X_t] &= \sum_{j=0}^{\infty} \psi_j E[\varepsilon_{t-j}] = 0, \\
E[X_t^2] &= \sum_{j=0}^{\infty} \psi_j^2 E[\varepsilon_{t-j}^2] = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 < \infty, \\
&\quad \text{as } \varepsilon_t \text{ are independent; we have constant finite variance.} \\
E[X_t \cdot X_{t-j}] &= E[(\varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots)(\varepsilon_{t-j} + \psi_1 \varepsilon_{t-j-1} + \psi_2 \varepsilon_{t-j-2} \\
&= \sigma^2 (\psi_j + \psi_{j+1}\psi_1 + \psi_{j+2}\psi_2 + \ldots) \\
&= \sigma^2 \sum_{k=0}^{\infty} \psi_k \psi_{k+j}, \qquad \text{depends on } j \text{ not } t.
\end{aligned}
$$

So we have a covariance stationary process in mean and variance.

ARMA Models

- Approximate Wold form with finite number of parameters.

  Wold Form:

$$Y_t - \mu = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \qquad \varepsilon_t \sim WN,$$

  ARMA(p,q):

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}.$$

Lag Operator

$$
\begin{aligned}
LX_t &\equiv X_{t-1}, \\
L^2X_t &= L \cdot LX_T = X_{t-2}, \\
L^{-1}X_t &\equiv X_{t+1}, \\
\Delta X_t &= (1-L)X_t = X_t - X_{t-1}.
\end{aligned}
$$

ARMA Models in Lag Notaion

ARMA(p,q):

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}.$$

With lag operator:

$$\begin{aligned}
\phi(L)(Y_t - \mu) &= \theta(L)\varepsilon_t, \\
\text{where} & \\
\phi(L) &= 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p, \\
\theta(L) &= 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q.
\end{aligned}$$

Stochastic Difference Equation (SDE) Representation

Let $X_t = Y_t - \mu$ and $w_t = \theta(L)\varepsilon_t$, then

$$\phi(L)X_t = w_t \text{ or } X_t = \theta_1 X_{t-1} + \ldots + \theta_p X_{t-p} + w_t,$$

is a p-order stochastic difference equation.

SDE Representation - AR(1) Example

Example: First-order SDE (AR(1)):

$$X_t = \phi X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN$$

- Solve for Wold Form (recursive substitution)

$$\begin{aligned} X_t &= \phi^{t+1} X_{-1} + \phi^t \varepsilon_0 + \phi^{t-1} \varepsilon_1 + \ldots + \phi \varepsilon_{t-1} + \varepsilon_t \\ &= \phi^{t+1} X_{-1} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}. \end{aligned}$$

  where X_{-1} is an initial condition and $\psi_i = \phi^i$.
- We approximated Wold form with 1 parameter form for AR(1).