Class 1: Stationary Time Series Analysis

Macroeconometrics - Fall 2009

Jacek Suda, BdF and PSE

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Outline

Outline:

1. Covariance-Stationary Processes
2. Wold Decomposition Theorem
3. ARMA Models
   1. Autoregressive Models
   2. Stability and Eigenvalues
   3. Moving Average Models
4. Auto-Correlation Function (ACF)
5. Partial Auto-Correlation Function (PACF)

Stochastic Process

- Stochastic process: a collection of random variables

$$\{\ldots, Y_{-1}, Y_0, Y_1, Y_2, \ldots, Y_T, \ldots\} = \{Y_t\}_{-\infty}^{\infty}.$$

- Observed series $\{y_1, y_2, \ldots, y_T\}$ – realizations of a stochastic process.

  We want a model for $\{Y_t\}_{\infty}^{-\infty}$ to explain observed realizations $\{y_t\}_1^T$.

Covariance-Stationary

Definition

$\{Y_t\}$ *is covariance-stationary (weak stationary) if*

(i) $E[Y_t] = \mu \quad \forall t$

(ii) $Cov(Y_t, Y_{t-j}) = E[(Y_t - \mu)(Y_{t-j} - \mu)] = \gamma_j, \quad \forall t, j$

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Note: mean is time-invariant

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Note: covariance doesn't depend on t

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Note: $Var(Y_t) = \gamma_0$ – variance is also constant.

Covariance-Stationary

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$\{Y_t\}$ *is covariance-stationary (weak stationary) if*

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(ii) $Cov(Y_t, Y_{t-j}) = E[(Y_t - \mu)(Y_{t-j} - \mu)] = \gamma_j, \quad \forall t, j$

- It is weak stationarity because it only relates to the first two moments. Higher moments can be time-variant.

Examples

1. $Y_t \sim iid(0, \sigma^2) \Rightarrow \{Y_t\}$ white noise (WN).
2. $Y_t \sim iidN(0, \sigma^2) \Rightarrow$ Gaussian white noise.

Strict (Strong) Stationary

Definition

$\{Y_t\}$ is (strictly/strongly) stationary if for any values of $j_1, j_2, \ldots, j_n$ the joint distribution of $(Y_t, Y_{t+j_1}, Y_{t+j_2}, \ldots, Y_{t+j_n})$ depends only on the intervals separating the dates $(j_1, j_2, \ldots, j_n)$ and not on date itself (t).

- For all $\tau, t_1, t_2, \ldots, t_n$:

$$F_Y(y_{t_1}, y_{t_2}, \ldots, y_{t_n}) = F_Y(y_{t_1+\tau}, y_{t_2+\tau}, \ldots, y_{t_n+\tau})$$

- If a process is strictly stationary with a finite second moment it is also covariance-stationary.
- Normality $\Rightarrow$ strong stationarity: whole distribution depends on the first two moments.

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$

Nonstationary Processes

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t - time dummy

Nonstationary Processes

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1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$

deterministic part

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$

stochastic component

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$
   - $E[Y_t] = \beta \cdot t$ depends on t
   - But $X_t = Y_t - \beta \cdot t$ is covariance stationary.

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$
   - $E[Y_t] = \beta \cdot t$ depends on t
   - But $X_t = Y_t - \beta \cdot t$ is covariance stationary.

2. $Y_t = Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN$, Y_0 constant
   - Random walk

Nonstationary Processes

Examples:

1. $Y_t = \beta \cdot t + \varepsilon_t, \qquad \varepsilon_t \sim WN$
   - $E[Y_t] = \beta \cdot t$ depends on t
   - But $X_t = Y_t - \beta \cdot t$ is covariance stationary.

2. $Y_t = Y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN, Y_0$ constant
   - Solving recursively :

$$Y_t = \sum_{j=1}^{t} \varepsilon_j + Y_0.$$

   - $E[Y_t] = Y_0$, time-invariant mean.
   - But $Var(Y_t) = t \cdot \sigma^2$ depends on t.
   - $X_t = Y_t - Y_{t-1}$ is covariance stationary.

Wold's Decomposition Theorem

Any covariance stationary $\{Y_t\}$ has infinite order, moving-average representation:

$$Y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} + \kappa_t, \qquad \psi_0 = 1, \varepsilon_t \sim WN.$$

- Linear combination of ε_s (innovations over time)
- Weights does not depend on time t, they only depend on j, i.e. how long ago the shock ε occurred.

Wold's Decomposition Theorem

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- $\sum_{j=0}^{\infty} \psi_j^2 < \infty$,
- $\varepsilon_t \sim WN(0, \sigma^2)$,
- κ_t deterministic term (perfectly forecastable).
- Example: $\kappa_t = \mu$, constant mean.

Wold's Decomposition Theorem - Illustration

Let $X_t = Y_t - \kappa_t$. Then,

$$
\begin{aligned}
E[X_t] &= \sum_{j=0}^{\infty} \psi_j E[\varepsilon_{t-j}] = 0, \\
E[X_t^2] &= \sum_{j=0}^{\infty} \psi_j^2 E[\varepsilon_{t-j}^2] = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 < \infty, \\
&\quad \text{as } \varepsilon_t \text{ are independent; we have constant finite variance.} \\
E[X_t \cdot X_{t-j}] &= E[(\varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots)(\varepsilon_{t-j} + \psi_1 \varepsilon_{t-j-1} + \psi_2 \varepsilon_{t-j-2} \\
&= \sigma^2 (\psi_j + \psi_{j+1}\psi_1 + \psi_{j+2}\psi_2 + \ldots) \\
&= \sigma^2 \sum_{k=0}^{\infty} \psi_k \psi_{k+j}, \qquad \text{depends on } j \text{ not } t.
\end{aligned}
$$

So we have a covariance stationary process in mean and variance.

ARMA Models

- Approximate Wold form with finite number of parameters.

  Wold Form:

$$Y_t - \mu = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \qquad \varepsilon_t \sim WN,$$

  ARMA(p,q):

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}.$$

Lag Operator

Define the operator L as

$$\begin{aligned} LX_t &\equiv X_{t-1}, \\ L^2X_t &= L \cdot LX_t = X_{t-2}. \end{aligned}$$

In general,

$$L^kX_t = X_{t-k}.$$

If c is a constant,

$$Lc = c.$$

Also,

$$\begin{aligned} L^{-1}X_t &\equiv X_{t+1}, \\ \Delta X_t &= (1-L)X_t = X_t - X_{t-1}. \end{aligned}$$

Lag Operator

It satisfies

$$\begin{array}{rcl} L(\alpha X_t + \beta Y_t) & = & \alpha X_{t-1} + \beta Y_{t-1} \\ (a\,L + b\,L^2)\,X_t & = & = aX_{t-1} + bX_{t-2}, \end{array}$$

and, when $|\phi| < 1$,

$$\lim_{j \to \infty} (1 + \phi L + \phi^2 L^2 + \ldots + \phi^j L^j) = (1 - \phi L)^{-1}$$

ARMA Models in Lag Notaion

- ARMA(p, q):

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q},$$

or

$$Y_t - \mu - \phi_1(Y_{t-1} - \mu) - \ldots - \phi_p(Y_{t-p} - \mu) = \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q},$$

- With lag operator:

$$\phi(L)(Y_t - \mu) = \theta(L)\varepsilon_t,$$

where

$$\begin{aligned} \phi(L) &= 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p, \\ \theta(L) &= 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q. \end{aligned}$$

Stochastic Difference Equation (SDE) Representation

Let $X_t = Y_t - \mu$ and $w_t = \theta(L)\varepsilon_t$.

Then

$$\phi(L)X_t = w_t,$$

or

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + w_t,$$

is a pth-order stochastic difference equation.

SDE Representation - AR(1) Example

Example: First-order SDE (AR(1)):

$$X_t = \phi X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN$$

- Solve for Wold Form (recursive substitution)

$$\begin{aligned} X_t &= \phi^{t+1} X_{-1} + \phi^t \varepsilon_0 + \phi^{t-1} \varepsilon_1 + \ldots + \phi \varepsilon_{t-1} + \varepsilon_t \\ &= \phi^{t+1} X_{-1} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}. \end{aligned}$$

  where X_{-1} is an initial condition and $\psi_i = \phi^i$.
- We approximated Wold form with 1 parameter form for AR(1).

Dynamic Multiplier

The dynamic multiplier measurers the effect of ε_t on subsequent values of X_τ:

$$\frac{\partial X_{t+j}}{\partial \varepsilon_t} = \frac{\partial X_j}{\partial \varepsilon_0} = \psi_j. \tag{1}$$

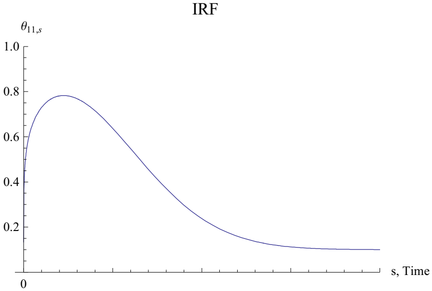
For the X_t being AR(1) process

$$\frac{\partial X_{t+j}}{\partial \varepsilon_t} = \psi_j = \phi^j. \tag{2}$$

The dynamic multiplier for any linear difference equations depends only on the length of time j, not on time t.

Impulse Response Function

The impulse-response function is a sequence of dynamic multipliers as a function of time from the one time impulse on ε_t



Cumulative impact

- Permanent increase in ε at time t, i.e. $\varepsilon_t = 1, \varepsilon_{t+1} = 1, \varepsilon_{t+2} = 1, \ldots$

$$\frac{\partial X_{t+j}}{\partial \varepsilon_t} + \frac{\partial X_{t+j}}{\partial \varepsilon_{t+1}} + \frac{\partial X_{t+j}}{\partial \varepsilon_{t+2}} + \ldots + \frac{\partial X_{t+j}}{\partial \varepsilon_{t+j}} = \psi_j + \psi_{j-1} + \ldots + \psi + 1$$

- In the limit, as $j \to \infty$

$$lim_{j\to\infty} \left[\frac{\partial X_{t+j}}{\partial \varepsilon_t} + \frac{\partial X_{t+j}}{\partial \varepsilon_{t+1}} + \ldots + \frac{\partial X_{t+j}}{\partial \varepsilon_{t+j}} \right] = \sum_{j=0}^{\infty} \psi_j = \psi(1),$$

where

$$\psi(1) = \psi(L = 1) = 1 + \psi_1 + \psi_2 + \ldots$$

AR(1) Model

Recall

$$
\begin{aligned}
X_t &= \phi X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN, \\
&= \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j} = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}.
\end{aligned}
$$

Wold coefficients

$$
\psi_j = \phi \psi_{j-1}, \qquad \psi_j = \frac{\partial Y_{t+j}}{\partial \varepsilon_t}
$$

- If $|\phi| < 1$ X_t is stationary solution to first-order SDE.
- If $\phi = 1$ then $\psi_j = 1 \; \forall j$ and $\quad X_t = X_{-1} + \sum_{j=0}^{t} \varepsilon_j$ is neither stationary nor stable solution, and $\psi(1)$ is infinite.

AR(1) lag notation

AR(1):

$$\begin{aligned} X_t &= \phi X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN, \\ (1-\phi L)X_t &= \varepsilon_t \end{aligned}$$

Multiply both sides by $(1-\phi L)^{-1}$:

$$\begin{aligned} X_t &= (1-\phi L)^{-1}\varepsilon_t \\ &= (1+\phi L+\phi^2L^2+\phi^3L^3+\ldots)\varepsilon_t \\ &= \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j} = \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}, \\ \psi(L) &= (1-\phi L)^{-1}. \end{aligned}$$

AR(1): Long-run effects

For AR(1), if $|\phi| < 1$

- the permanent increase in ε_t equals

$$\frac{\partial X_{t+j}}{\partial \varepsilon_t} + \frac{\partial X_{t+j}}{\partial \varepsilon_{t+1}} + \ldots + \frac{\partial X_{t+j}}{\partial \varepsilon_{t+j}} = 1 + \phi + \phi^2 + \phi^3 + \ldots + \phi^j.$$

  and as $j \to \infty$

$$\psi(1) = 1 + \phi + \phi^2 + \ldots = 1/(1-\phi)$$

- the cumulative consequences for X of a one-time change in ε,

$$\sum_{j=0}^{\infty} \frac{\partial X_{t+j}}{\partial \varepsilon_t} = 1/(1-\phi)$$

AR(1) Mean

- Intercept representation for $Y_t \equiv X_t + \mu$

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t, \qquad \text{where } c = \mu(1-\phi).$$

- Mean

$$E[Y_t] = c + \phi E[Y_{t-1}] + E[\varepsilon_t],$$

Since we have covariance stationary process, $E[Y_t] = E[Y_{t-1}]$ and

$$E[Y_t] = \frac{c}{1-\phi} \equiv \mu.$$

AR(1) Variance

- Variance

$$
\begin{aligned}
Var(Y_t) &= E[(Y_t - \mu)^2] \\
&= E[(\phi(Y_{t-1} - \mu) + \varepsilon_t)^2] \\
&= \phi^2 E[(Y_{t-1} - \mu)^2] + 2\phi E[(Y_{t-1} - \mu)\varepsilon_t] + E[\varepsilon_t^2].
\end{aligned}
$$

Since Y_t is covariance stationary and ε_t is independently distributed,

$$Var(Y_t) = Var(Y_{t-1})$$

and

$$E[Y_{t-1}\varepsilon_t] = 0,$$

so

$$Var(Y_t) = \frac{\sigma^2}{1 - \phi^2} \equiv \gamma_0.$$

AR(1): Covariance

- Covariance

$$
\begin{aligned}
Cov(Y_t, Y_{t-j}) &= E[(Y_t - \mu)(Y_{t-j} - \mu)] \\
&= \phi E[(Y_{t-1} - \mu)(Y_{t-j} - \mu)] + E[\varepsilon_t (Y_{t-j} - \mu)], \\
Cov(Y_t, Y_{t-j}) &\equiv \gamma_j = \phi \gamma_{j-1}.
\end{aligned}
$$

AR(1): Moments

- Summing up

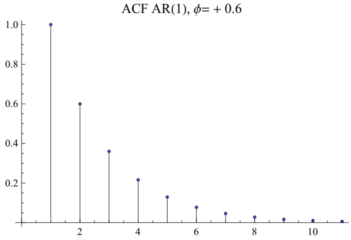
$$\begin{aligned} E[Y_t] &= \frac{c}{1-\phi} \equiv \mu, \\ Var(Y_t) &= \frac{\sigma^2}{1-\phi^2} \equiv \gamma_0, \\ Cov(Y_t, Y_{t-j}) &\equiv \gamma_j = \phi\gamma_{j-1}. \end{aligned}$$

AR(1): Auto-correlation Function (ACF)

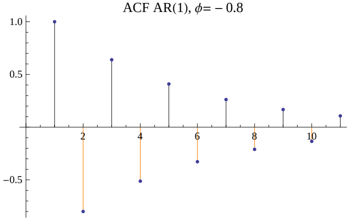
- Define
$$\rho_j \equiv \frac{\gamma_j}{\gamma_0} \equiv j^{\text{th}} \text{ autocorrelation } \equiv corr(Y_t, Y_{t-j})$$
- For AR(1), $\rho_j = \phi\rho_{j-1}$.
  - For AR(1) ACF and IRF are the same. In general it not true.
  - ACF $\in < -1, 1 >$.

AR(1): Auto-correlation Function (ACF)

- $0 < \phi < 1$



- $-1 < \phi < 0$



*P*th-order SDE: AR(p)

An AR(p) process

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + v_t,$$

can be rewritten as a 1^{st} order system

$$\begin{bmatrix} Y_t - \mu \\ Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ \vdots \\ Y_{t-p+1} - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ Y_{t-3} - \mu \\ \vdots \\ Y_{t-p} - \mu \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In the state-space, companion form notation

$$\beta_t = F \cdot \beta_{t-1} + \varepsilon_t$$

and we are back in 1^{st} order system.

Pth-order SDE: AR(p)

An AR(p) process

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An AR(p) process

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can be rewritten as a 1^{st} order system

$$\begin{bmatrix} Y_t - \mu \\ Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ \vdots \\ Y_{t-p+1} - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ Y_{t-3} - \mu \\ \vdots \\ Y_{t-p} - \mu \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In the state-space, companion form notation

$$\beta_t = F \cdot \beta_{t-1} + \varepsilon_t$$

and we are back in 1^{st} order system.

Pth-order SDE: AR(p)

An AR(p) process

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + v_t,$$

can be rewritten as a 1^{st} order system

$$\begin{bmatrix} Y_t - \mu \\ Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ \vdots \\ Y_{t-p+1} - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & \ldots & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ Y_{t-3} - \mu \\ \vdots \\ Y_{t-p} - \mu \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} Y_t - \mu \\ Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ \vdots \\ Y_{t-p+1} - \mu \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & .. & 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1} - \mu \\ Y_{t-2} - \mu \\ Y_{t-3} - \mu \\ \vdots \\ Y_{t-p} - \mu \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In the state-space, companion form notation

$$\beta_t = F \cdot \beta_{t-1} + \varepsilon_t$$

and we are back in 1^{st} order system.

AR(p): Stability

Consider a state space form

$$\beta_{t+j} = F^{j+1}\beta_{t-1} + F^{j}\varepsilon_t + \ldots + F\varepsilon_{t+j-1} + \varepsilon_{t+j}.$$

AR(p) is stable and stationary if

$$\lim_{j\to\infty} F^{j} = 0$$

,

- i.e. when eigenvalues of F are inside unit circle (have modulus < 1).
- Shocks die out.

Eigenvalues

- Consider equation

$$Fx = \lambda x.$$

x is eigenvector and λ is a corresponding eigenvalue.
To compute the eigenvalue, write it as

$$(F - \lambda \mathbb{I})x = 0.$$

If x is a non-zero vector then

$$F - \lambda \mathbb{I} \text{ is singular} \qquad \Rightarrow \qquad \det(F - \lambda \mathbb{I}) = 0.$$

Example

Consider the AR(2)

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + v_t$$

Then, in a matrix notation

$$\beta_t = F\beta_{t-1} + \varepsilon_t,$$

where

$$\beta_t = \begin{bmatrix} Y_t - \mu \\ Y_{t-1} - \mu \end{bmatrix}, \qquad F = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues λ of the matrix F solves

$$\Rightarrow \det \begin{pmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda\phi_1 - \phi_2 = 0,$$

$$\lambda_i = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

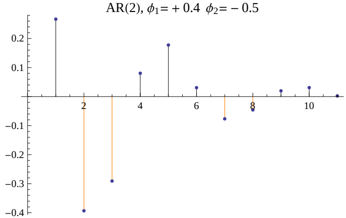
If $|\lambda_i|<1$ for $i = 1, 2$, the AR(2) is stable .

AR(p): Stability

- For pth-order SDE, solve

$$\lambda^p - \phi_1 \lambda^{p-1} - \ldots - \phi_{p-1}\lambda - \phi_p = 0$$

- In general, the solution involve complex and real roots.
- The AR(p) system is stable if all eigenvalues are inside the unit circle. Note: complex eigenvalues imply periodic behavior.



Characteristic Equation

Recall that we could write down AR(p) process

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) + \varepsilon_t,$$

in the lag polynomial form

$$\phi(L)(Y_t - \mu) = \varepsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p.$$

The stability can be then studied through the characteristic equation

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0.$$

Roots of Characteristic Equation

- The roots (z's) of characteristic equation, $\phi(z) = 0$, are inverse of eigenvalues (λ's) of companion matrix F:

$$z = 1/\lambda.$$

as

$$\begin{aligned} 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p &= (1 - \lambda_1 z)(1 - \lambda_2 z) \cdots (1 - \lambda_p z) \\ &= z^p (1/z - \lambda_1)(1/z - \lambda_2) \cdots (1/z - \lambda_p) \end{aligned}$$

- For $|z| > 1$ stochastic difference equation is stable and stationary.

Stability

$\phi(L)$ can be decomposed into

$$(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) = (1 - \lambda_1 L)(1 - \lambda_2 L)\ldots(1 - \lambda_p L)$$

Factor the polynomial into AR(1) elements.

- Since $\forall i \; |\lambda_i| < 1$ then $(1 - \lambda_i L)^{-1}$ exists and so $\phi(L)^{-1}$ exists

$$\Longrightarrow \qquad \psi(L) = \phi(L)^{-1} \qquad \text{for AR(p).}$$

Yule-Walker Equations

Variance, covariances, autocorrelations, and dynamic multipliers have the same p^{th}-order form:

$$
\begin{aligned}
Var(Y_t) = \gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_2 + \ldots + \phi_p\gamma_p + \sigma^2 \\
Cov(Y_t, Y_{t-j}) = \gamma_j &= \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + \ldots + \phi_p\gamma_{j-p} \\
corr(Y_t, Y_{t-j}) = \rho_j &= \phi_1\rho_{j-1} + \phi_2\rho_{j-2} + \ldots + \phi_p\rho_{j-p} \\
\psi_j &= \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \ldots + \phi_p\psi_{j-p}
\end{aligned}
$$

Yule-Walker Equations: AR(2) Example

- AR(2) Example:

$$\begin{gathered}\phi(L)^{-1} = \psi(L) \Rightarrow \phi(L)\psi(L) = 1 \\ (1 - \phi_1 L - \phi_2 L^2)(1 + \psi_1 L + \psi_2 L^2 + \dots) = 1 \\ 1 + (\psi_1 - \phi_1)L + (\psi_2 - \phi_1\psi_1 + \phi_2)L^2 + \dots = 1\end{gathered}$$

Then

$$\begin{aligned}\psi_1 &= \phi_1, \\ \psi_2 &= \phi_1\psi_1 + \phi_2, \\ &\vdots \\ \psi_j &= \phi_1\psi_{j-1} + \phi_2\psi_{j-2} \quad .\end{aligned}$$

AR(2) Process:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$$

- Mean

$$\mu = c/(1 - \phi_1 - \phi_2)$$

- Covariance

$$\begin{aligned}
\gamma_j &= E[(Y_t - \mu)(\phi_1(Y_{t-j-1} - \mu) + \phi_2(Y_{t-j-2} - \mu) + \varepsilon_t)] \\
&= \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \\
\gamma_1 &= \phi_1 \gamma_0 + \phi_2 \gamma_1 \\
&= \gamma_0 \frac{\phi_1}{1 - \phi_2} \\
\gamma_2 &= \phi_1 \gamma_1 + \phi_2 \gamma_0 \\
&= \gamma_0 \left(\frac{\phi_1^2}{1 - \phi_2} + \phi_2 \right)
\end{aligned}$$

- Variance

$$
\begin{aligned}
\gamma_0 &= E[(Y_t-\mu)(\phi_1(Y_{t-1}-\mu)+\phi_2(Y_{t-2}-\mu)+\varepsilon_t)] \\
&= \phi_1\gamma_1+\phi_2\gamma_2+\sigma^2 \\
&= \left[\frac{\phi_1^2}{1-\phi_2}+\frac{\phi_2\phi_1^2}{1-\phi_2}+\phi_2^2\right]\gamma_0+\sigma^2 \\
\gamma_0 &= \frac{1-\phi_2}{(1+\phi_2)\left[(1-\phi_2)^2-\phi_1^2\right]}
\end{aligned}
$$

- Autocorrelation

$$
\begin{aligned}
\rho_j &= \phi_1\rho_{j-1}+\phi_2\rho_{j-2} \\
\rho_1 &= \phi_1+\phi_2\rho_1 \\
&= \phi_1/(1-\phi_2) \\
\rho_2 &= \phi_1\rho_1+\phi_2
\end{aligned}
$$

Moving Average (MA) Processes

Recall Wold Form:

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \qquad \varepsilon_t \sim WN,$$

MA(q) process : truncated Wold form.

$$\begin{aligned} Y_t &= \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q} \\ Y_t &= \mu + \theta(L)\varepsilon_t, \\ & \theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q. \end{aligned}$$

- "Moving average" as Y_t is constructed from a weighed sum to the q most recent values of ε

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1 + \theta L)\varepsilon_t$$

Moments

- Mean

$$E[Y_t] = \mu,$$

- Variance

$$Var(Y_t) = \gamma_0 = E[(Y_t - \mu)^2]$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1 + \theta L)\varepsilon_t$$

Moments

- Mean

$$E[Y_t] = \mu,$$

- Variance

$$\begin{aligned} Var(Y_t) = \gamma_0 &= E[(Y_t - \mu)^2] \\ &= E[(\varepsilon_t + \theta\varepsilon_{t-1})^2] \end{aligned}$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1 + \theta L)\varepsilon_t$$

Moments

- Mean

$$E[Y_t] = \mu,$$

- Variance

$$\begin{aligned} Var(Y_t) = \gamma_0 &= E[(Y_t - \mu)^2] \\ &= E[(\varepsilon_t + \theta\varepsilon_{t-1})^2] \\ &= E[\varepsilon_t^2 + 2\theta\varepsilon_t\varepsilon_{t-1} + \theta^2\varepsilon_{t-1}^2] \end{aligned}$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1+\theta L)\varepsilon_t$$

Moments

- Mean

$$E[Y_t] = \mu,$$

- Variance

$$\begin{aligned} Var(Y_t) = \gamma_0 &= E[(Y_t - \mu)^2] \\ &= E[(\varepsilon_t + \theta\varepsilon_{t-1})^2] \\ &= E[\varepsilon_t^2 + 2\theta\varepsilon_t\varepsilon_{t-1} + \theta^2\varepsilon_{t-1}^2] \\ &= \sigma^2 + 0 + \theta^2\sigma^2 \end{aligned}$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1 + \theta L)\varepsilon_t$$

Moments

- Mean

$$E[Y_t] \quad = \quad \mu,$$

- Variance

$$\begin{aligned} Var(Y_t) = \gamma_0 &= E[(Y_t - \mu)^2] \\ &= E[(\varepsilon_t + \theta\varepsilon_{t-1})^2] \\ &= E[\varepsilon_t^2 + 2\theta\varepsilon_t\varepsilon_{t-1} + \theta^2\varepsilon_{t-1}^2] \\ &= \sigma^2 + 0 + \theta^2\sigma^2 \\ &= (1 + \theta^2)\sigma^2, \end{aligned}$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1 + \theta L)\varepsilon_t$$

Moments

- Covariance

$$Cov(Y_t, Y_{t-1}) \quad = \quad \gamma_1 = E[(Y_t - \mu)(Y_{t-1} - \mu)]$$

- Autocorrelation Function

$$\begin{aligned} \rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma_2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2} \\ \rho_j &= 0, \qquad \forall j > 1. \end{aligned}$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1+\theta L)\varepsilon_t$$

Moments

- Covariance

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= \gamma_1 = E[(Y_t - \mu)(Y_{t-1} - \mu)] \\ &= E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})] \end{aligned}$$

- Autocorrelation Function

$$\begin{aligned} \rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma_2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2} \\ \rho_j &= 0, \qquad \forall j > 1. \end{aligned}$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1+\theta L)\varepsilon_t$$

Moments

- Covariance

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= \gamma_1 = E[(Y_t - \mu)(Y_{t-1} - \mu)] \\ &= E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})] \\ &= \theta\sigma^2, \end{aligned}$$

- Autocorrelation Function

$$\begin{aligned} \rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma_2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2} \\ \rho_j &= 0, \qquad \forall j > 1. \end{aligned}$$

MA(1)

Example MA(1)

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} = \mu + (1 + \theta L)\varepsilon_t$$

Moments

- Covariance

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= \gamma_1 = E[(Y_t - \mu)(Y_{t-1} - \mu)] \\ &= E[(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2})] \\ &= \theta\sigma^2, \\ Cov(Y_t, Y_{t-j}) &= \gamma_j = 0 \qquad \forall j > 1. \end{aligned}$$

- Autocorrelation Function

$$\begin{aligned} \rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{\theta\sigma_2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2} \\ \rho_j &= 0, \qquad \forall j > 1. \end{aligned}$$

Invertibility of MA Processes

Consider MA(1)

$$Y_t = \mu + (1 + \theta L)\varepsilon_t$$

- θ and $1/\theta$ give the same ACF

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1+\theta^2} = \frac{1/\theta}{1+\frac{1}{\theta}} = \frac{1}{\theta}\frac{1}{\frac{\theta^2+1}{\theta^2}} = \frac{\theta}{1+\theta^2}$$

Normalize θ by using invertible MA representation.

- Example: Both

$$Y_t = \varepsilon_t + 0.5\varepsilon_{t-1}$$

and

$$Y_t = \varepsilon_t + 2\varepsilon_{t-1}$$

have the same autocorrelation function:

$$\rho_1 = \frac{2}{1+4} = \frac{0.5}{1+(0.5)^2} = 0.4$$

Invertibility of MA Processes

But for MA(1) with $|\theta| < 1$, there exists ∞-order AR representation:

$$\begin{aligned} Y_t - \mu &= (1 + \theta L)\varepsilon_t, \qquad |\theta| < 1, \\ &= (1 - \theta^* L)\varepsilon_t, \qquad \text{for } \theta^* = -\theta \end{aligned}$$

It can rewritten as

$$\begin{aligned} (1 - \theta^* L)^{-1}(Y_t - \mu) &= \varepsilon_t \\ (1 + \theta^* L + {\theta^*}^2 L^2 + \ldots)(Y_t - \mu) &= \varepsilon_t \\ Y_t = \theta(Y_{t-1} - \mu) - \theta^2(Y_{t-2} - \mu) &+ \theta^3(Y_{t-3} - \mu) \ldots + \varepsilon_t \end{aligned}$$

i.e. AR(∞) with $\phi(L) = 1 + \theta - \theta^2 + \theta^3 - \theta^4 + \ldots$.

$|\theta| < 1$ is not a stability requirement, MA system is always stable. It allows invertibility and AR(∞) representation.

Partial Autocorrelation Function (PACF)

Definition

k^{th}-order partial autocorrelation is regression coefficient (for the population) $\phi_k k$ in k^{th}-order autoregression

$$Y_t = c + \phi_{k1} Y_{t-1} + \phi_{k2} Y_{t-2} + \ldots + \phi_{kk} Y_{t-k} + \varepsilon_t$$

It measures how important is the last lag in the process.