

Lecture: Solving Linear DSGE Models

We have a set of linear expectational equations:

$$AE_t x_{t+1} + Bx_t + Cv_{t+1} = 0. \quad (\text{A})$$

We seek a solution of the form

$$x_{t+1} = Fx_t + Gv_{t+1}. \quad (\text{B})$$

This solution represents the time series behavior of $\{x_t\}$ as a function of $\{v_t\}$, where v_t is a vector of exogenous innovations, or as frequently referenced, structural shocks.

In this note we use a standard timing for capital, that is

$$y_t = z_t k_t^\alpha l_t^{1-\alpha}$$

for production function,

$$k_{t+1} = (1 - \delta)k_t + i_t$$

for capital accumulation equation,

$$Div_t = y_t - w_t l_t - r_t k_t$$

for profit/dividend function, and

$$c_t + i_t = w_t l_t + r_t k_t + Div_t$$

for the budget constraint.

The linearized equation with these equations are then as follows.

Equation 1:

$$\sigma(E_t \hat{c}_{t+1} - \hat{c}_t) = (1 - \beta(1 - \delta))E_t \hat{r}_{t+1} \quad (1)$$

Equation 2:

$$\varphi \hat{l}_t + \sigma \hat{c}_t = \hat{w}_t \quad (2)$$

Equation 3:

$$\hat{k}_{t+1} = (1 - \delta)k_t + \frac{i^{ss}}{k^{ss}} \hat{i}_t \quad (3)$$

Equation 4:

$$\left(1 - \frac{i^{ss}}{y^{ss}}\right) \hat{c}_t + \frac{i^{ss}}{y^{ss}} \hat{i}_t = \hat{y}_t \quad (4)$$

Equation 5:

$$\hat{y}_t = \hat{z}_t + \alpha \hat{k}_t + (1 - \alpha) \hat{l}_t \quad (5)$$

Equation 6:

$$\hat{y}_t - \hat{l}_t = \hat{w}_t \quad (6)$$

Equation 7:

$$\hat{y}_t - \hat{k}_t = \hat{r}_t \quad (7)$$

Equation 8:

$$\hat{z}_t = \rho \hat{z}_{t-1} + \epsilon_t \quad (8)$$

Our original system has 8 equations but to we can reduce the system by eliminating some variables/equations.

First, equations (2) and (6) gives

$$\hat{l}_t = \frac{1}{1 + \varphi} \hat{y}_t - \frac{\sigma}{1 + \varphi} \hat{c}_t$$

Putting equation (5) and expression for \hat{l}_t yields

$$\begin{aligned} \hat{y}_t &= z_t + \alpha \hat{k}_t + (1 - \alpha) \left(\frac{1}{1 + \varphi} \hat{y}_t - \frac{\sigma}{1 + \varphi} \hat{c}_t \right) \\ \left(1 - \frac{1 - \alpha}{1 + \varphi}\right) \hat{y}_t &= z_t + \alpha \hat{k}_t - \frac{(1 - \alpha)\sigma}{1 + \varphi} \hat{c}_t \\ \hat{y}_t &= \frac{1 + \varphi}{\alpha + \varphi} z_t + \frac{1 + \varphi}{\alpha + \varphi} \alpha \hat{k}_t - \frac{(1 - \alpha)\sigma}{\alpha + \varphi} \hat{c}_t \end{aligned}$$

Taking from equation (4) formula for \hat{i}_t and putting it into equation (3) yields

$$\hat{k}_{t+1} = (1 - \delta)k_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t - \frac{c^{ss}}{k^{ss}}\hat{c}_t$$

and using expression for \hat{y}_t ,

$$\begin{aligned}\hat{k}_{t+1} - (1 - \delta)k_t &= -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t \\ \hat{k}_{t+1} - (1 - \delta)k_t &= -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\left(\frac{1 + \varphi}{\alpha + \varphi}\hat{z}_t + \frac{1 + \varphi}{\alpha + \varphi}\alpha\hat{k}_t - \frac{(1 - \alpha)\sigma}{\alpha + \varphi}\hat{c}_t\right)\end{aligned}$$

Equation (1) can be written as

$$\begin{aligned}\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) &= \beta r^{ss} E_t \hat{r}_{t+1} \\ \sigma(E_t\hat{c}_{t+1} - \hat{c}_t) &= \beta r^{ss} E_t \hat{y}_{t+1} - \beta r^{ss} E_t \hat{k}_{t+1}\end{aligned}$$

which, using expression for y_t above, can be written as

$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = -\beta r^{ss} E_t \hat{k}_{t+1} + \beta r^{ss} \frac{1 + \varphi}{\alpha + \varphi} E_t \hat{z}_{t+1} + \beta r^{ss} \frac{1 + \varphi}{\alpha + \varphi} \alpha E_t \hat{k}_{t+1} - \beta r^{ss} \frac{(1 - \alpha)\sigma}{\alpha + \varphi} E_t \hat{c}_{t+1}$$

or

$$-\sigma\hat{c}_t + \sigma\left(1 + \beta r^{ss} \frac{(1 - \alpha)}{\alpha + \varphi}\right) E_t \hat{c}_{t+1} = \beta r^{ss} \left(\frac{1 + \varphi}{\alpha + \varphi} \alpha - 1\right) E_t \hat{k}_{t+1} + \beta r^{ss} \frac{1 + \varphi}{\alpha + \varphi} E_t \hat{z}_{t+1}. \quad (9)$$

Note that together with equation (8)

$$\hat{z}_t = \rho\hat{z}_{t-1} + \epsilon_t$$

we have transform our original set of equations into the system of 3 equations with 3 endogenous variables (c, k, z), and 1 exogenous variable (ϵ).

We can write it as

$$\kappa_{11}\hat{c}_t + \kappa_{12}E_t\hat{k}_{t+1} + \kappa_{13}E_t\hat{z}_{t+1} + \kappa_{14}E_t\hat{c}_{t+1} = 0 \quad (10)$$

$$\kappa_{21}\hat{c}_t + \kappa_{22}\hat{k}_t + \kappa_{23}\hat{z}_t + \kappa_{24}\hat{k}_{t+1} = 0 \quad (11)$$

$$\hat{z}_{t+1} + \rho\hat{z}_t = \epsilon_{t+1} \quad (12)$$

where κ_{ij} contain the ‘deep’ parameters of the model.

Setting $x_t = \{k_t, c_t\}$ and $v_t = z_t$ the model has the form consistent with equation (A):

$$AE_t x_{t+1} + Bx_t + Cv_{t+1} = 0.$$

with A , B , and C being functions of κ s.

(Alternatively, we could set $x_t = \{z_t, k_t, c_t\}$ and $v_t = \epsilon_t$ and solve the model.)

1 Blanchard and Kahn’s Method

Blanchard and Kahn(1980) developed the method to solve models written as

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + E f_t, \quad (13)$$

where the model’s variables have been divided into an $n \times 1$ vector of endogenous pre-determined variables x_{1t} (defined as variables for which $E_t x_{1t+1} = x_{1t+1}$), and an $m \times 1$ vector of endogenous non-predetermined variables x_{2t} . The $k \times 1$ vector f_t contains exogenous forcing variables.

For our model to conform to this specification we need to pre-multiply the entire system by A^{-1} so that

$$E_t x_{t+1} = -A^{-1}Bx_t - A^{-1}Cv_{t+1}.$$

which is possible only if A is invertible. Then $\tilde{A} = -A^{-1}B$ and $E = -A^{-1}C\rho$.

In $x_t = \{\hat{c}_t, \hat{k}_t\}$, \hat{k}_t is predetermined (given \hat{k}_{t-1} and $\hat{i}_{t-1} = \hat{y}_{t-1} - \hat{c}_{t-1}$, \hat{k}_t is deter-

mined by (3)); \hat{c}_t is endogenous but not predetermined (as indicated in (9) or (10)); and \hat{z}_t is an exogenous forcing variable. Thus in the notation of (13), we seek a specification of the model in the form

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t \hat{c}_{t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + E \hat{z}_t. \quad (14)$$

To solve this system, Blanchard and Khan start with a Jordan decomposition to obtain Jordan normal form of \tilde{A} :

$$\tilde{A} = \Lambda^{-1} J \Lambda. \quad (15)$$

where J is a diagonal matrix with the eigenvalues of \tilde{A} along its leading diagonal and zeros in the off-diagonal elements, and Λ is a matrix of the corresponding eigenvectors. Eigenvalues are ordered in increasing absolute value in moving from left to right.¹ In particular, J can be written as

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \quad (16)$$

where the eigenvalues in J_1 lie on or within the unit circle (i.e. of modulus less than one), and those in J_2 lie outside of the unit circle. J_2 is said to be unstable or explosive, since J_2^n diverges as n increases.

Before we proceed, we partition matrices Λ and E

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad (17)$$

where Λ_{11} is conformable with J_1 , etc.

Blanchard-Khan condition

The Blanchard-Khan condition states that if the number of explosive eigenvalues is

¹Eigenvalues of a matrix M are obtained from the solution of equations of the form $Me = \lambda e$, where e is an eigenvector and λ the associated eigenvalue.

equal to the number of non-predetermined variables, the system is said to be saddle-path stable and a unique solution to the model exists.

On the other hand, if the number of explosive eigenvalues (i.e. greater than 1) exceeds the number of non-predetermined variables then no solution exists (and the system is said to be a source). Finally, if the number of explosive eigenvalues is less than the number of non-predetermined variables, then an infinity of solutions exist (and the system is said to be a sink).

If Blanchard-Khan condition holds then under the Jordan decomposition of \tilde{A} , (13) yields

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \Lambda^{-1} J \Lambda \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} f_t. \quad (18)$$

Pre-multiplying by Λ we obtain

$$\begin{bmatrix} \bar{x}_{1t+1} \\ E_t \bar{x}_{2t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} f_t. \quad (19)$$

where $\bar{x}_t = \Lambda x_t$ and $D = \Lambda E$, that is

$$\begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \quad (20)$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \quad (21)$$

Under this transformation the two sets of equations are now decoupled, so that the non-predetermined variables depend only on the unstable eigenvalues of \tilde{A} contained in J_2 , as expressed in the lower part of (19). In other words, we can write each time $t + 1$ variable as solely a function of predetermined variables, exogenous variables and controls at time t .

To find the solution for the non-predetermined variables, we iterate forward lower

portion of (19) as follows. Since

$$E_t \bar{x}_{2t+1} = J_2 \bar{x}_{2t} + D_2 f_{2t}$$

we can write \bar{x}_{2t} as

$$\bar{x}_{2t} = J_2^{-1} E_t \bar{x}_{2t+1} - J_2^{-1} D_2 f_{2t}. \quad (22)$$

Iterating it forward

$$\bar{x}_{2t+1} = J_2^{-1} E_{t+1} \bar{x}_{2t+2} - J_2^{-1} D_2 f_{2t+1}, \quad (23)$$

and substituting into (22) yields²

$$\bar{x}_{2t} = J_2^{-2} E_t \bar{x}_{2t+2} - J_2^{-2} D_2 E_t f_{2t+1} - J_2^{-1} D_2 f_{2t}. \quad (24)$$

Since J_2 contains eigenvalues that are outside the unit root, $\lim_{n \rightarrow \infty} J_2^{-n} = 0$. Therefore, iterating the substitution gives the following expression

$$\bar{x}_{2t} = - \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i}. \quad (25)$$

Using equation (20),

$$(\quad \bar{x}_{2t} = \Lambda_{21} x_{1t} + \Lambda_{22} x_{2t}, \quad)$$

we can map it back to x_{2t} as

$$x_{2t} = -\Lambda_{22}^{-1} \Lambda_{21} x_{1t} - \Lambda_{22} \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i}. \quad (26)$$

In the case of our model, $E_t(f_{2t+i}) = \rho^i \hat{z}_t$, and thus (26) becomes

$$x_{2t} = -\Lambda_{22}^{-1} \Lambda_{21} x_{1t} - \Lambda_{22} J_2^{-1} (I - \rho J_2^{-1})^{-1} D_2 \hat{z}_t. \quad (27)$$

To solve the second part of the system we use the upper portion of (18)

$$\left(\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} f_t. \right)$$

²We use the Law of Iterated Expectations: $E_t[E_{t+1}(x_{t+2})] = E_t(x_{t+2})$ for any x_t .

$$x_{1t+1} = \tilde{A}_{11}x_{1t} + \tilde{A}_{12}x_{2t} + E_1f_t, \quad (28)$$

where $\tilde{A} = \Lambda^{-1}J\Lambda$, and \tilde{A}_{11} , \tilde{A}_{12} are conformable with x_{1t} and x_{2t} . Plugging the expression for x_{2t} in equation (26) we obtain a solution for x_{1t} .

Note that the key requirement for this method to be applicable is the invertibility of the matrix A in

$$AE_t x_{t+1} + Bx_t + Cv_t = 0.$$

2 Sims's Method

An alternative (but similar) approach to solving the linear expectation systems was developed by Sims (2001). He proposes a solution method applied to models expressed as

$$Ax_t = Bx_{t-1} + Cv_t + D\eta_t + E, \quad (29)$$

where E is a vector of constants. Instead of expressing variables in terms of expected values, Sims drops expectations operators and introduces the expectations errors, η_{t+1} , defined as $x_{t+1} = E_t x_t + \eta_{t+1}$. Additionally, the exogenous shock is now incorporated into x_t (that is z_t is a part of x_t).³

Sim's method involves following steps.

Step 1:

Instead of Jordan decomposition, use 'QZ factorization' (generalized Schur decomposition) to decompose A and B as

$$A = Q'\Lambda Z' \quad (30)$$

$$B = Q'\Omega Z' \quad (31)$$

where (Q, Z) are unitary, and (Λ, Ω) are upper triangular.⁴ Next, (Q, Z, Λ, Ω) are or-

³Note that our model needs to be lagged by one period in order to match the notation (and code) of Sims.

⁴A unitary matrix Q satisfies $Q'Q = QQ' = I$. If Q and/or Z contain complex values, the transpositions reflect complex conjugation, that is, each complex entry is replaced by its conjugate and then transposed.

dered such that, in absolute value, the generalized eigenvalues of A and B are organized in Λ and Ω in increasing order moving from left to right. These generalized eigenvalues, ϑ , of A and B are obtained as the solution to $Ae = \vartheta Be$, and ϑ can be calculated as the ratio of diagonal elements of A and B , $\vartheta_i = \frac{\lambda_{ii}}{\omega_{ii}}$.⁵

Using these factorization, pre-multiply both sides of the equation by Q . The original system can be then expressed as

$$\Lambda z_t = \Omega z_{t-1} + QCv_t + QD\eta_t + QE, \quad (32)$$

where $z_{t+1} = Z'x_{t+1}$.⁶

Step 2:

Partition equation (32) into explosive and non-explosive blocks:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [Cv_t + D\eta_t + E]. \quad (33)$$

The model can now be solved iteratively.

Step 3:

To solve the lower (explosive) part of the system, z_{2t} , note that the lower block of (33) can be written as

$$\Lambda_{22}z_{2t} = \Omega_{22}z_{2t-1} + w_{2t}, \quad (34)$$

where $w_t = Q[Cv_t + D\eta_t + E]$ and w_{1t} and w_{2t} are partitioned conformably. Using equation (34) we can compute z_{2t} as

$$z_{2t} = M_{22}z_{2t+1} - \Omega_{22}^{-1}w_{2t+1}, \quad (35)$$

where $M = \Omega_{22}^{-1}\Lambda_{22}$. Iterating forward yields the expression for z_{2t}

$$z_{2t} = - \sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} w_{2t+1+i}, \quad (36)$$

⁵Sims' website provides a program that computes and orders the eigenvalues appropriately.

⁶Since Q is unitary $QQ' = I$.

since $\lim_{t \rightarrow \infty} M^t z_{2t} = 0$. (Recall that $M = \Omega_{22}^{-1} \Lambda_{22}$ and generalized eigenvalues expressed as diagonal elements λ_{ii}/ω_{ii} are all outside unit circle in the lower block of the system.)

Since w_t is defined as $w_t = Q[Cv_t + D\eta_t + E]$, equation (36) expresses z_{2t} as a function of future values of structural and expectational errors.

Since z_{2t} is known at time t , and $E_t(\eta_{t+s}) = E_t(v_{t+s}) = 0$ for $s > 0$ equation (36) can be written as

$$z_{2t} = - \sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} Q_2 E_2, \quad (37)$$

where $Q_2 E_2$ are the lower portion of QE conformable with z_2 .⁷

Since $-\sum_{i=0}^{\infty} M^i = -(I - M)^{-1}$, the solution of z_{2t} is obtained as

$$z_{2t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 E_2. \quad (38)$$

Step 4:

The final step is to solve for z_{1t} in equation (33). Note that the solution of z_{1t} requires a solution for the expectations errors that appear in (33).

Sims notes that if there is a unique solution for the model, there exist a systematic relationship exists between the expectations errors associated with z_{1t} and z_{2t} .

Necessary and sufficient condition for uniqueness

If there exists a $k \times (n - k)$ matrix Φ that satisfies

$$Q_1 D = \Phi Q_2 D, \quad (39)$$

the equilibrium is unique.

Φ represents the systematic relationship between the expectations errors associated with z_{1t} and z_{2t} noted above.

If there exists a unique equilibrium we can calculate Φ , as in equation (39), and the

⁷Sims also considers the case in which the structural innovations v_t are serially correlated, which leads to a generalization of (37).

solution of z_{1t} can be computed. If we pre-multiply (32)

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} [Cv_t + D\eta_t + E].$$

by $[I - \Phi]$, we obtain

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 - \Phi Q_2 \\ Q_2 \end{bmatrix} [Cv_t + D\eta_t + E]. \quad (40)$$

If equation (39) holds, the loading factor for the expectational errors, η_t , is zero. Then the system may be written in the form

$$x_t = \Theta_e + \Theta_0 x_{t-1} + \Theta_1 v_t, \quad (41)$$

where

$$H = Z \begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12} - \Phi\Lambda_{22}) \\ 0 & I \end{bmatrix} \quad (42)$$

$$\Theta_e = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1} Q_2 \end{bmatrix} E \quad (43)$$

$$\Theta_0 = Z \Lambda_{11}^{-1} [\Omega_{11} (\Omega_{12} - \Phi \Omega_{22})] Z' \quad (44)$$

$$\Theta_1 = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ 0 \end{bmatrix} D \quad (45)$$