

# Wykład: Model nakładających się pokoleń (OverLapping Generation model)

Makroekonomia / Macroeconomics  
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Michał Brzoza-Brzezina & Jacek Suda

# Introduction

- OLG models are used in variety of applications.
- They have features that make them distinct:
  - competitive equilibria need not be Pareto Optimal,
  - Ricardian equivalence does not hold.

⇒ There is a role for government intervention.
- Agents in the model are heterogeneous with respect to
  - age,
  - (possibly) individual productivity.

# Basic 2-period OLG model

- Time is discrete and continues forever:  $t = 0, 1, 2 \dots$
- In each period,  $L_t \in \mathbb{R}^+$  two-period-lived agents are born.
- An agent is endowed with one unit of labor in the first period and with zero units of labor in the second period.  
(agents can work only when young)
- Population evolves according to  $L_t = (1 + n)^t L_0$  .  
(if  $n = 0$  no population growth)

# Agents

- At date 0 there are some *old* agents who live for one period and are collectively endowed with  $K_0 \in \mathbb{R}^+$  units of capital.
- An agent born at date  $t$  has preferences represented by

$$u(c_t^y, c_{t+1}^o).$$

- $c_t^y$  denotes the consumption of a young agent at date  $t$ .
- $c_t^o$  denotes the consumption of an old agent at date  $t$ .
- $u(\cdot)$  is utility function that is strictly increasing in both arguments, strictly concave, and continuously differentiable.

# Utility

- Define the ratio

$$v(c^y, c^o) \equiv \frac{u_1(c^y, c^o)}{u_2(c^y, c^o)}.$$

- Then

$$\lim_{c^y \rightarrow 0} v(c^y, c^o) = \infty, \quad \text{for any } c^o > 0,$$

$$\lim_{c^o \rightarrow 0} v(c^y, c^o) = 0, \quad \text{for any } c^y > 0.$$

- The initial old seek to maximize consumption at date 0.

# Investment technology and capital

- Consumption goods can be converted one-for-one into capital and vice versa.
- Capital constructed at date  $t$  becomes productive only at date  $t + 1$ .
- There is no capital depreciation ( $\delta = 0$ ).
- Young agents sell their labor to firms and save in the form of capital accumulation.
- Old agents rent capital to firms.
- After production takes place, old agents convert capital into goods for consumption.

# Firms

- Representative firm rents capital and hires labor to produce goods.
- Technology is represented by the production function

$$Y_t = F(K_t, L_t),$$

- $Y_t$  represents output,
- $K_t$  denotes capital input,
- $L_t$  denotes labor input,
- $F(\cdot, \cdot)$  is strictly increasing in each argument, strictly quasiconcave, twice differentiable, and homogeneous of degree one.

# Young Consumer's Problem

- An agent born at date  $t$  solves the problem

$$\max_{c_t^y, c_{t+1}^o, s_t} u(c_t^y, c_{t+1}^o)$$

subject the budget constraints

$$c_t^y + s_t \leq w_t,$$

$$c_{t+1}^o \leq (1 + r_{t+1})s_t,$$

- $s_t$  is saving when young,
  - $w_t$  is the wage rate at date  $t$ ,
  - $r_{t+1}$  is the capital rental rate at date  $t + 1$ .
- Consumer chooses savings and consumption when young and old treating prices,  $w_t$  and  $r_{t+1}$ , as being fixed.
- Agents have perfect foresight (i.e., they can perfectly predict prices): at time  $t$  the consumer is assumed to know  $r_{t+1}$ .

- Substituting for  $c_t^y$  and  $c_{t+1}^o$  from budget constraints reduces the maximization problem to the choice of one variable ( $s_t$ ):

$$\max_{s_t} u(w_t - s_t, (1 + r_{t+1})s_t).$$

- The first-order condition for an optimum is

$$-u_1(w_t - s_t, (1 + r_{t+1})s_t) + u_2(w_t - s_t, (1 + r_{t+1})s_t)(1 + r_{t+1}) = 0.$$

- It determines the optimal savings,  $s_t$ , as a function of prices

$$s_t = s(w_t, r_{t+1}).$$

- We can also write the FOC as

$$\frac{u_1}{u_2} = 1 + r_{t+1}$$

(the intertemporal marginal rate of substitution equals one plus the interest rate)

- If consumption when young and consumption when old are both normal goods, then  $\frac{\partial s}{\partial w_t} > 0$ .
- The same does not have to be true for  $\frac{\partial s}{\partial r_{t+1}}$  due to opposing income and substitution effects.

# Representative Firm's Problem

- Representative firm solves a one period, static problem

$$\max_{K_t, L_t} [F(K_t, L_t) - w_t L_t - r_t K_t].$$

- The first-order conditions for a maximum profits are

$$F_K(K_t, L_t) - r_t = 0,$$

$$F_L(K_t, L_t) - w_t = 0.$$

- Since the production function is homogenous of degree one, we can define

$$f(k_t) \equiv F(k_t, 1) = F(K_t/L_t, 1),$$

where  $k_t = K_t/L_t$  is capital per worker (not per capita).

- Then,

$$r_t = f'(k_t),$$

$$w_t = f(k_t) - k_t f'(k_t).$$

# Definition: Competitive Equilibrium

## Definition

A *competitive equilibrium* is a sequence of quantities  $\{k_{t+1}, s_t\}_{t=0}^{\infty}$  and prices  $\{w_t, r_t\}_{t=0}^{\infty}$  which satisfy

- consumer's optimization,  $(s_t = s(w_t, r_{t+1}))$ ;
- firm's optimization,  $(r_t = f'(k_t)$  and  $w_t = f(k_t) - k_t f'(k_t)$ ); and
- market clearing,  $((1+n)k_{t+1} = s(w_t, r_{t+1}))$

at any date  $t \geq 0$  given the initial capital-labor ratio  $k_0$ .

# Characterization of Equilibrium

- Market-clearing condition for the capital rental market is given by

$$K_{t+1} = L_t s(w_t, r_{t+1}).$$

- We can rewrite it as

$$\frac{L_{t+1} K_{t+1}}{L_t L_{t+1}} = s(w_t, r_{t+1}) \quad \Rightarrow \quad (1+n)k_{t+1} = s(w_t, r_{t+1}).$$

- As  $r_t = f'(k_t)$  and  $w_t = f(k_t) - k_t f'(k_t)$  it can be written as

$$(1+n)k_{t+1} = s(f(k_t) - k_t f'(k_t), f'(k_{t+1})).$$

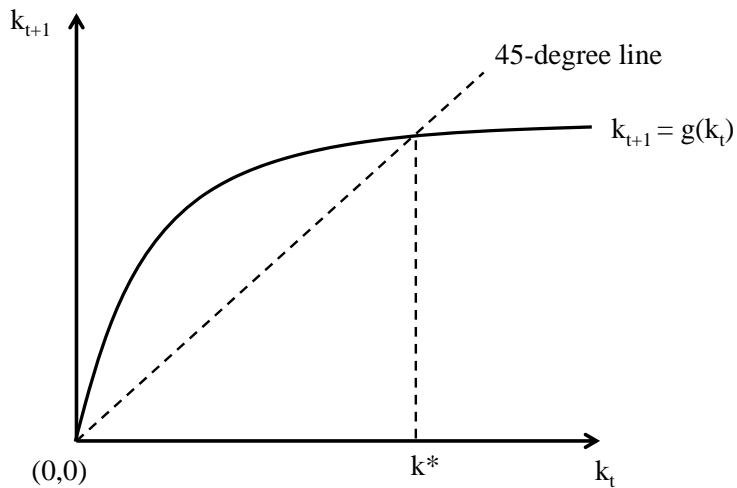
- Given  $k_0$ , the equilibrium evolution of the capital-labor ratio is given by

$$(1 + n)k_{t+1} = s(f(k_t) - k_t f'(k_t), f'(k_{t+1})).$$

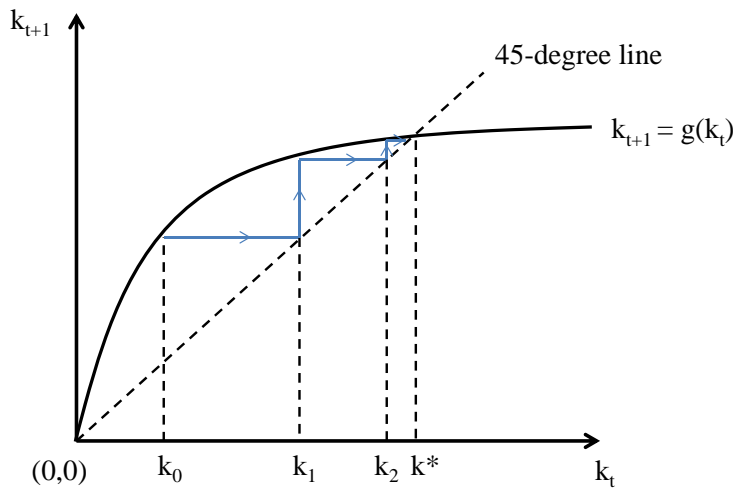
- It is a nonlinear first-order difference equation (solution  $k_{t+1} = g(k_t)$ ).
- Given the equilibrium sequence of the capital-labor ratio, we can recover the equilibrium prices from

$$\begin{aligned}r_t &= f'(k_t), \\w_t &= f(k_t) - k_t f'(k_t).\end{aligned}$$

# Figure 1: Dynamic System



# Figure 1: Dynamic System



# Definition: Optimal Allocation

## Definition

A *Pareto optimal allocation* is a sequence  $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$  satisfying resource constraint

$$f(k_t) + k_t = (1 + n)k_{t+1} + c_t^y + \frac{c_t^o}{1 + n}$$

$$\left( \text{ or } \quad F(K_t, L_t) + K_t = K_{t+1} + c_t^y L_t + c_t^o L_{t-1}, \quad \right)$$

and with the property that there exists no other feasible allocation  $\{\hat{c}_t^y, \hat{c}_t^o, \hat{k}_{t+1}\}_{t=0}^{\infty}$  satisfying

$$\begin{aligned} \hat{c}_t^o &\geq c_t^o \\ u(\hat{c}_t^y, \hat{c}_t^o) &\geq u(c_t^y, c_t^o) \end{aligned}$$

for all  $t \geq 0$ , with strict inequality in at least one instance.

# Optimal Allocation

- A Pareto optimal allocation is a feasible allocation such that there is no other feasible allocation for which all consumers are at least as well off and some consumer is better off.
- To characterize Pareto optimal allocations solve the social planner's problem who has control over production, capital accumulation, and the distribution of consumption goods between the young and the old.

# Optimal Allocation

- Solve the planner's problem to characterize Pareto optimal allocations.
- For simplicity, suppose

$$u(c_t^y, c_{t+1}^o) = h(c_t^y) + \beta h(c_{t+1}^o)$$

with  $h$  increasing, strictly concave, and continuously differentiable.

- Suppose, for now, that the planner cares only about the utility of the current and  $T$  future generations.

# Social Planner Problem

- Define the social welfare function

$$W_T = \beta h(c_0^o) + \sum_{t=0}^{T-1} [h(c_t^y) + \beta h(c_{t+1}^o)]$$

- Planner chooses  $\{c_0^o, c_0^y, k_1, c_1^o, c_1^y, k_2, \dots, c_{T-1}^o, c_{T-1}^y, k_T, c_T^o\}$  to maximize social welfare subject to resource constraint

$$f(k_t) + k_t = (1+n)k_{t+1} + c_t^y + \frac{c_t^o}{1+n}$$

and terminal condition for  $k_{T+1}$ .

- After substituting  $c_t^y$  from the resource constraint, the FOCs to the social planner problem are

$$\partial W_T / \partial c_t^o : \quad \beta h'(c_t^o) - \frac{h'(c_t^y)}{1+n} = 0$$

$$\partial W_T / \partial k_{t+1} : \quad -(1+n)h'(c_t^y) + [f'(k_{t+1}) + 1]h'(c_{t+1}^y) = 0$$

for all  $t = 0, \dots, T-1$ .

# Stationary Solution

- Consider a stationary solution to the planner's problem:

$$c_t^o = c^o, \quad c_t^y = c^y, \quad k_{t+1} = k \quad \text{for all } t = 0, 1, \dots, T - 1.$$

- A stationary allocation  $(c^y, c^o, k)$  satisfies

$$(1 + n)\beta h'(c^o) = h'(c^y),$$

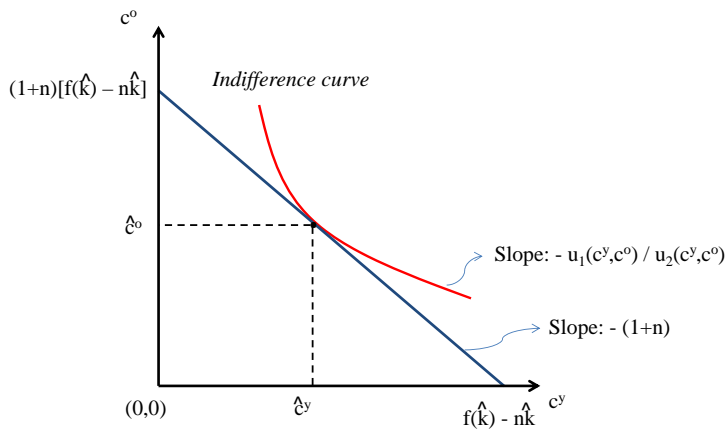
$$f'(k) = n$$

and

$$f(k) - nk = c^y + \frac{c^o}{1 + n}$$

- Stationary solution remains the same for  $T \rightarrow \infty$ .

# Figure 2: Pareto Optimum



# Example

- Suppose  $h(c) = \ln(c)$  and  $F(K, L) = \gamma K^\alpha L^{1-\alpha}$  with  $0 < \alpha < 1$  and  $\gamma > 0$ .
- Young agent solves

$$\max_{s_t} [\ln(w_t - s_t) + \beta \ln([1 + r_{t+1}]s_t)],$$

which yields an optimal saving function

$$s_t = \frac{\beta}{1 + \beta} w_t$$

- The FOCs for the firm's profit maximization problem give

$$r_t = \gamma \alpha k_t^{\alpha-1}$$

$$w_t = \gamma (1 - \alpha) k_t^\alpha$$

$k^*$ 

- The FOCs for a firm and young agent problem, together with  $k_{t+1}(1+n) = s_t$ , give equilibrium law of motion for capital accumulation

$$k_{t+1}(1+n) = \frac{\beta}{1-\beta} \gamma(1-\alpha)k_t^\alpha.$$

- In this case, for a given  $k_0$  we obtain a unique equilibrium sequence for capital  $\{k_t\}_{t=0}^\infty$ .
- In the competitive equilibrium, the steady state is given by

$$k^* = \left[ \frac{\beta}{1-\beta} \frac{\gamma(1-\alpha)}{1+n} \right]^{\frac{1}{1-\alpha}}.$$

$\hat{k}$ 

- We can compute the steady-state for the planner problem.
- For our functional assumptions, the social planner allocation satisfied

$$(1+n)\beta h'(c^o) = h'(c^y),$$

$$f'(k) = n,$$

which, for our functional assumptions, equals

$$(1+n)\beta c^y = c^o,$$

$$\gamma \alpha k^{1-\alpha} = n.$$

- Socially optimal steady state capital stock is

$$\hat{k} = \left[ \frac{\gamma \alpha}{n} \right]^{\frac{1}{1-\alpha}}.$$

# Dynamic in/efficiency

- In general,  $k^* \neq \hat{k}$ :
  - ⇒ the competitive equilibrium steady state is in general not socially optimal
- Economy may suffer from a dynamic inefficiency:
  - depending on parameters, there might be too much or too little capital in the steady.
- Suppose  $\beta = 1$  and  $n = 0.3$ :
  - if  $0 < \alpha < 0.103$  then  $k^* > \hat{k}$ , and
  - if  $0.103 < \alpha < 1$  then  $\hat{k} > k^*$ .
- Inefficiency due to  $k^* \neq \hat{k}$ :
  - ① quantity of resources available to allocate between the young and the old is not optimal,
  - ②  $r^* \neq n$  so consumers misallocate consumption goods over time:
    - ⇒ too much or too little saving in a competitive equilibrium .

# Government Debt

- Government debt can be used to introduce intergenerational transfers.
- Government issues debt to young agents, transfers the proceeds to old agents, and then taxes the young of the next generation in order to pay the interest and principal on the debt.
- $B_{t+1}$  is the quantity of one-period bonds issued by the government at date  $t$  (and bought back in period  $t + 1$ ).
- Bond is a promise to pay  $1 + r_{t+1}$  units of consumption goods at date  $t + 1$ .
- Interest rate on government bonds is the same as the rental rate of capital:
  - government bonds and capital are perfect substitutes in equilibrium.
- Assume  $B_{t+1} = bL_t$  with  $b > 0$  fixed: the quantity of government debt is fixed in per-capita terms.

# Taxes

- Quantity of government debt is fixed in per-capita terms.
- Taxes  $T_t$  are levied lump-sum on young agents.
- Government's budget constraint is

$$B_{t+1} + T_t = (1 + r_t)B_t,$$

i.e. the revenues from new bond issues and taxes in period  $t$ ,  $T_t$ , equals the payments of interest and principal on government bonds issued in period  $t - 1$ .

- Let  $\tau_t$  denote the lump-sum tax per young agent

$$T_t = \tau_t L_t.$$

- Solving for  $\tau_t$  yields

$$\tau_t = \frac{T_t}{L_t} = \frac{(1 + r_t)B_t - B_{t+1}}{L_t} = \left( \frac{r_t - n}{1 + n} \right) b.$$

# Young Consumer's Problem

- Young agent solves

$$\max_{s_t} u(w_t - s_t - \tau_t, (1 + r_{t+1})s_t).$$

- Agents save in the form of capital and government bonds.
- Optimal savings for a young agent is given by

$$s_t = s(w_t - \tau_t, r_{t+1}).$$

- As before, profit maximization by the firm implies

$$\begin{aligned} r_t &= f'(k_t), \\ w_t &= f(k_t) - k_t f'(k_t). \end{aligned}$$

# Definition: Competitive Equilibrium with Government Debt

## Definition

A *competitive equilibrium* is a sequence of quantities  $\{k_{t+1}, s_t, \tau_t\}_{t=0}^{\infty}$  and prices  $\{w_t, r_t\}_{t=0}^{\infty}$  which satisfy

- consumer's optimization,  $(s_t = s(w_t - \tau_t, r_{t+1}))$ ;
- firm's optimization,  $(r_t = f'(k_t)$  and  $w_t = f(k_t) - k_t f'(k_t)$ );
- market clearing,  $((1 + n)k_{t+1} + b = s(w_t - \tau_t, r_{t+1}))$ ; and
- government budget constraint,  $(\tau_t = \left(\frac{r_t - n}{1+n}\right) b)$ ,

at any date  $t \geq 0$  given the initial capital-labor ratio  $k_0$ .

# Characterization of Equilibrium with Government Debt

- Using market clearing condition, equilibrium law of motion for the capital-labor ratio is given by

$$(1+n)k_{t+1} + b = s(f(k_t) - k_t f'(k_t) - \left(\frac{f'(k_t) - n}{1+n}\right) b, f'(k_{t+1})) \quad .$$

- As before, given  $k_0$ , we obtain a unique equilibrium sequence  $\{k_t\}_{t=0}^{\infty}$ .
- Steady state capital-labor ratio  $k^*$  satisfies therefore

$$(1+n)k^* + b = s(f(k^*) - k^* f'(k^*) - \left(\frac{f'(k^*) - n}{1+n}\right) b, f'(k^*)) \quad .$$

- Note that we have  $k^* = k^*(b)$  (i.e., capital-labor ratio depends on the government's fiscal policy  $b$ ).

# Optimal level of $b$

- Since  $k^*$  depends on fiscal policy, social planner can use government bond to restore/implement Pareto optimal allocation.
- Choose the fiscal policy  $b$  such that the competitive equilibrium steady state is socially optimal.
- In other words, find  $\hat{b}$  such that  $k^*(\hat{b}) = \hat{k}$ .
- Because  $f(\hat{k}) = n$ , it follows that

$$\hat{b} = -(1+n)\hat{k} + s(f(\hat{k}) - n\hat{k}, n) \quad .$$

- Note that  $\hat{b}$  can be positive or negative.

# Optimal level of $\tau_t$

- In the steady state with  $b = \hat{b}$ ,  $\tau_t = 0$  and, hence,  $T_t = 0$ .
- ⇒ Debt increases at a rate just sufficient to pay the interest and principal on previously issued debt.
- Optimum government debt policy transfers wealth:
    - from the young to the old if the debt is positive, or
    - from the old to the young if the debt is negative .

# Example

- Consider as before  $u(c^y, c^o) = \ln c^y + \beta \ln c^o$  and  $F(K, L) = \gamma K^\alpha L^{1-\alpha}$ .
- Optimal savings for a young agent is

$$s_t = \left( \frac{\beta}{1 + \beta} \right) (w_t - \tau_t).$$

- The equilibrium sequence  $\{k_t\}_{t=0}^\infty$  is determined by

$$k_{t+1}(1 + n) + b = \left( \frac{\beta}{1 + \beta} \right) \left[ (1 - \alpha)\gamma k_t^\alpha - \frac{(\alpha\gamma k_t^{1-\alpha} - n)b}{1 + n} \right].$$

- The steady-state capital-labor ratio  $k^*(b)$  solves the equation above with  $k^*(b) = k_t = k_{t+1}$ .
- The level of debt that brings Pareto-optimal level of steady-state capital-labor ratio is given by

$$\begin{aligned} \hat{b} &= \left( \frac{\beta}{1 + \beta} \right) (1 - \alpha)\gamma \left( \frac{\alpha\gamma}{n} \right)^{\frac{\alpha}{1-\alpha}} - \left( \frac{\alpha\gamma}{n} \right)^{\frac{1}{1-\alpha}} (1 + n) \\ &= \gamma \left( \frac{\alpha\gamma}{n} \right)^{\frac{\alpha}{1-\alpha}} \left[ \frac{\beta(1 - \alpha)}{1 + \beta} - \frac{\alpha}{n} \right]. \end{aligned}$$

# Ricardian equivalence

- Note that government debt matters.
- Ricardian equivalence does not hold here because the taxes required to pay off the currently-issued debt are not levied on the agents who receive the current tax benefits from a higher level of debt today.
- Here, government debt policy implements the intergenerational transfers that are required to achieve optimality.

# MONEY IN THE OLG MODEL

# OLG model with Money

- Time is discrete and continues forever:  $t = 1, 2, 3 \dots$
- In each period,  $L_t$  two-period-lived agents are born.
- In period 1 there are  $L_0$  *old* agents who live for one period.
- Population evolves according to  $L_t = (1 + n)^t L_0$ , with  $n > -1$ .  
(if  $n = 0$  no population growth)

# Agents

- An agent born at date  $t$  has preferences represented by

$$u(c_t^t, c_{t+1}^t)$$

- $c_t^t$  denotes the consumption at date  $t$  of an agent born in period  $t$  (*young agent*).
- $c_{t+1}^t$  denotes the consumption at date  $t + 1$  of an agent born in period  $t$  (*old agent*).
- $u(\cdot, \cdot)$  is utility function that is strictly increasing in both arguments, strictly concave, twice continuously differentiable, and for

$$v(c_y, c_o) \equiv \frac{\frac{\partial u(c_y, c_o)}{\partial c_y}}{\frac{\partial u(c_y, c_o)}{\partial c_o}}.$$

we have

$$\begin{aligned} \lim_{c_y \rightarrow 0} v(c_y, c_o) &= \infty, & \text{for any } c_o > 0, \\ \lim_{c_o \rightarrow 0} v(c_y, c_o) &= 0, & \text{for any } c_y > 0, \end{aligned}$$

- agents want to consume positive amounts in both periods of life.

# Money

- An agent receives  $y$  units of perishable consumption good in the first period (when young), and nothing in the second period (when old),
  - agent's endowment profile is  $e = (e_t^t, e_t^{t+1}) = (y, 0)$ .
- $L_0$  initial old agents are collectively endowed with  $M_0$  units of fiat money.
- Money is perfectly divisible, intrinsically useless, and can not be privately produced.
- Money are injected/withdrawn through lump-sum transfers to old agents in each period,  $\tau_t$ .

# Money

- Total money supply in period  $t$  equals  $M_t$  and the government budget constraint is

$$p_t(M_t - M_{t-1}) = L_{t-1}\tau_t,$$

- $p_t$  denotes the price of money in terms of the period  $t$  consumption good,
  - $p_t = 0$  implies the case where money has no value.
- Money supply grows at a constant rate

$$M_t = zM_{t-1} \quad z \geq 0.$$

- The government budget constraint becomes

$$p_t M_t \left(1 - \frac{1}{z}\right) = L_{t-1} \tau_t.$$

# Social Planner

- Consider a social planner that can re-distribute young agents' endowments across the population.
- The resource constraint is given by

$$L_t c_t^t + L_{t-1} c_{t-1}^t \leq L_t y, \quad t = 1, 2, \dots$$

- in each period, total consumption of the young ( $L_t c_t^t$ ) plus total consumption of the old ( $L_{t-1} c_{t-1}^t$ ) can not exceed the total endowment ( $L_t y$ ).
- Assume planner chooses a stationary allocations, i.e.  $(c_t^t, c_t^{t+1}) = (c_y, c_o)$ , for all  $t$ .
- The resource constraint becomes

$$c_y + \frac{c_o}{1+n} \leq y$$

and we call any allocation  $(c_y, c_o)$  that satisfies this equation *feasible*.

# Definition: Pareto optimal allocation

## Definition

A *Pareto optimal allocation*, chosen from the class of stationary allocations,  $(c_y, c_o)$ , is feasible and satisfies the property that there exists no other feasible stationary allocation  $(\hat{c}_y, \hat{c}_o)$  such that

$$\begin{aligned}u(\hat{c}_y, \hat{c}_o) &\geq u(c_y, c_o), \\ \hat{c}_o &\geq c_o\end{aligned}$$

with at least one of these two inequalities being a strong inequality.

# Stationary social planner optimal allocation

- The stationary solution to the planner problem solves

$$\max_{c_y, c_o} u(c_y, c_o)$$

subject to

$$c_y + \frac{c_o}{1+n} = y.$$

- The interior solution  $(c_y^*, c_o^*)$  satisfies

$$-u_1(c_y^*, c_o^*) + (1+n)u_2(c_y^*, c_o^*) = 0.$$

# Young Consumer's Problem

- An agent born at date  $t$  solves the problem

$$\max_{c_t^t, c_{t+1}^t, m_t} u(c_t^t, c_{t+1}^t)$$

subject to budget constraints

$$c_t^t + p_t m_t \leq y,$$

$$c_{t+1}^t \leq p_{t+1} m_t + \tau_{t+1},$$

- $m_t$  is a nominal quantity of money acquired by *young* agent,
  - $\tau_{t+1}$  is money transfers from government to old agents at date  $t + 1$ ,
- Agents choose money holdings,  $m_t$ , and consumption treating sequence of prices  $p_t, p_{t+1}$  and money transfers  $\tau_t$  as given.

# Definition: Competitive Equilibrium

## Definition

A *competitive equilibrium* is a sequence of consumption allocations  $\{(c_t^t, c_{t+1}^t)\}_{t=1}^{\infty}$ , a sequence of individual money demands  $\{m_t\}_{t=1}^{\infty}$ , a sequence of money supplies  $\{M_t\}_{t=1}^{\infty}$ , a sequence of money transfers  $\{\tau_t\}_{t=1}^{\infty}$ , and as sequence of prices  $\{p_t\}_{t=1}^{\infty}$ , given  $M_0$ , such that

- ①  $(c_t^t, c_{t+1}^t)$  and  $m_t$  maximize agents' utility subject to budget constraints, given  $p_t, p_{t+1}$ , and  $\tau_t$ ;
- ② money supply grows at a constant rate  $z$ ;
- ③ government budget constraint holds; and
- ④ markets clear,  $(p_t M_t = L_t p_t m_t)$ .

for all  $t = 1, 2, \dots$

# Non-monetary equilibrium

- In this economy, there always exists a non-monetary equilibrium,  
 $\Rightarrow$  a competitive equilibrium where  $p_t = 0$  for all  $t$ , (money has no value) .
- In non-monetary equilibrium

$$(c_t^t, c_{t+1}^t) = (y, 0) \quad \text{and} \quad \tau_t = 0 \quad \text{for all } t.$$

- For  $p_t = 0$ , this allocation
  - is feasible,
  - solves utility maximization of an agent,
  - clears the asset (money) and goods markets.
- This allocation is, however, Pareto dominated by allocation  $(c_y^*, c_o^*)$ :
  - $u(c_y^*, c_o^*) \geq u(y, 0)$ ,
  - $c_o^* > 0$ .

# Absence of double-coincidence-of-wants

- In the absence of money, there is no trade.
- Recall that endowment profile is  $(y, 0)$ .
- There is absence-of-double-coincidence of wants:
  - an agent born in period  $t$  wants to trade period  $t$  consumption goods for period  $t + 1$  consumption goods, but
  - no agent wants to trade period  $t + 1$  consumption goods for period  $t$  consumption goods.

# Monetary equilibria

- Consider equilibria with  $p_t > 0$  for all  $t$ .
- Young agents solve

$$\max_{c_t^t, c_{t+1}^t, m_t} u(c_t^t, c_{t+1}^t)$$

subject to

$$c_t^t + p_t m_t \leq y \quad c_{t+1}^t \leq p_{t+1} m_t + \tau_{t+1}.$$

taking  $p_t$  and  $\tau_t$  as given.

- This can be rewritten as

$$\max_{m_t} u(y - p_t m_t, p_{t+1} m_t + \tau_{t+1}).$$

- The first-order condition for an interior optimum is

$$-p_t u_1(y - p_t m_t, p_{t+1} m_t + \tau_{t+1}) + p_{t+1} u_2(y - p_t m_t, p_{t+1} m_t + \tau_{t+1}) = 0$$

where  $u_i(c_t^t, c_{t+1}^t) = \frac{\partial u(c_t^t, c_{t+1}^t)}{\partial c_{t+i}^t}$ .

# Real quantity of money

- Let  $q_t$  denote real per generation quantity of money:

$$q_t = \frac{p_t M_t}{L_t} .$$

- Money market clearing becomes

$$q_t = p_t m_t .$$

- Government budget constraint,  $L_t \tau_{t+1} = p_{t+1} (M_{t+1} - M_t)$ , implies

$$c_t^{t+1} = p_{t+1} m_t + \tau_{t+1} = p_{t+1} \left( \frac{M_t}{L_t} + \frac{M_{t+1}}{L_t} - \frac{M_t}{L_t} \right) = (1+n)q_{t+1} .$$

- Moreover, as

$$\frac{p_{t+1} M_t}{L_t} = \frac{p_{t+1} M_{t+1}}{L_{t+1}} \frac{M_t}{M_{t+1}} \frac{L_{t+1}}{L_t} = \frac{1+n}{z} q_t ,$$

the first-order condition for young agents can be written as

$$-q_t u_1(y - q_t, (1+n)q_{t+1}) + q_{t+1} \frac{1+n}{z} u_2(y - q_t, (1+n)q_{t+1}) = 0 .$$

# Monetary equilibria

- This is a first-order difference equation in  $q_t$ .
- Its solution yield the equilibrium sequence of real per generation quantity of money,  $\{q_t\}_{t=1}^{\infty}$ .
- Using

$$p_t = \frac{q_t N_t}{M_t} ,$$

we can solve for price sequence  $\{p_t\}_{t=1}^{\infty}$ .

- The sequence of money transfers  $\{\tau_t\}_{t=1}^{\infty}$  is given by

$$\tau_t = (1 + n)q_t \left(1 - \frac{1}{z}\right)$$

while the sequence of consumptions can be determined from

$$(c_t^t, c_{t+1}^t) = (y - q_t, (1 + n)q_{t+1}) .$$

# Stationary monetary equilibrium

- Consider a stationary monetary equilibrium, i.e. competitive equilibrium in which

$$q_t = q \quad \text{for all } t.$$

- The equilibrium can be obtained by solving the first-order condition,

$$u_1(y - q, (1 + n)q) = \frac{1 + n}{z} u_2(y - q, (1 + n)q) .$$

- If  $z = 1$ , i.e. money supply is constant, this equation equals

$$u_1(c_y^*, c_o^*) = (1 + n)u_2(c_y^*, c_o^*)$$

with  $c_y^* = y - q$  and  $c_o^* = (1 + n)q$ .

⇒ The stationary monetary equilibrium under fixed money supply is Pareto optimal.

# Inflation

- Using equation

$$p_t = \frac{q_t N_t}{M_t},$$

we can determine the gross inflation rate as

$$\pi_{t+1} = \frac{p_t}{p_{t+1}} = \frac{q_t}{q_{t+1}} \frac{z}{1+n}.$$

- In the stationary equilibrium

$$\pi^* = \frac{z}{1+n}$$

and the equilibrium is Pareto optimal regardless of the inflation rate.

- In fact, for any  $z \leq 1$ , stationary monetary equilibrium is Pareto optimal.

# Example 1

- Assume that  $u(c_y, c_o) = \ln c_y + \ln c_o$ .
- The condition for monetary equilibrium

$$-q_t u_1(y - q_t, (1+n)q_{t+1}) + q_{t+1} \frac{1+n}{z} u_2(y - q_t, (1+n)q_{t+1})$$

becomes

$$\frac{q_t}{y - q_t} = \frac{1}{z}.$$

- The solution is

$$q_t = \frac{y}{1+z}.$$

- The stationary monetary equilibrium is the unique equilibrium.
- The consumption allocations are

$$(c_t^t, c_{t+1}^t) = \left( \frac{zy}{1+z}, \frac{(1+n)y}{1+z} \right)$$

$c_t^t$  increases with money growth rate;  $c_{t+1}^t$  decreases with  $z$ .

## Example 2

- Assume that  $u(c_y, c_o) = c_y^{\frac{1}{2}} + c_o^{\frac{1}{2}}$ .
- The condition for monetary equilibrium for this utility equals

$$q_{t+1} = \frac{q_t^2 z^2}{(y - q_t)(1 + n)} .$$

- There are multiple solution to this equation.
- The stationary monetary equilibrium is

$$q_t = \frac{(1 + n)y}{1 + n + z^2} .$$

- There exists also a continuum of equilibria indexed by  $q_1$ ,

$$q_1 \in \left( 0, \frac{(1 + n)y}{1 + n + z^2} \right) ,$$

such that

$$\lim_{t \rightarrow \infty} q_t = 0$$

- nonstationary monetary equilibria converging to the nonmonetary equilibrium