Lecture: Solving Linear DSGE Models

Makroekonomia / Macroeconomics
SD SGH

Jacek Suda

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Linear Expectational Equations

We consider a set of linear expectational equations of the form:

$$AE_t x_{t+1} + Bx_t + Cv_{t+1} = 0. \tag{A}$$

We seek a solution of the form:

$$x_{t+1} = Fx_t + Gv_{t+1}. \tag{B}$$

This solution represents the time-series behavior of $\{x_t\}$ as a function of $\{v_t\}$, where v_t is a vector of exogenous innovations, or structural shocks.

Example: The RBC Model Timing

In this note, we use a standard timing for capital:

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

for the production function,

$$k_{t+1} = (1-\delta)k_t + i_t$$

for the capital accumulation equation,

$$Div_t = y_t - w_t l_t - r_t k_t$$

for the profit/dividend function, and

$$c_t + i_t = w_t l_t + r_t k_t + Div_t$$

for the budget constraint.

Linearized RBC Model (Equations 1–4)

The linearized equilibrium conditions are:

1. Euler Equation:

$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = (1 - \beta(1-\delta))E_t\hat{r}_{t+1} \tag{1}$$

2. Labor Supply:

$$\varphi\hat{l}_t + \sigma\hat{c}_t = \hat{w}_t \tag{2}$$

3. Capital Accumulation:

$$\hat{k}_{t+1} = (1-\delta)k_t + \frac{i^{ss}}{k^{ss}}\hat{i}_t \tag{3}$$

4. Resource Constraint:

$$(1 - \frac{i^{ss}}{y^{ss}})\hat{c}_t + \frac{i^{ss}}{y^{ss}}\hat{i}_t = \hat{y}_t \tag{4}$$

Linearized RBC Model (Equations 5–8)

5. Production Function:

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_t + (1-\alpha)\hat{l}_t \tag{5}$$

6. Labor Demand:

$$\hat{y}_t - \hat{l}_t = \hat{w}_t \tag{6}$$

7. Capital Demand:

$$\hat{y}_t - \hat{k}_t = \hat{r}_t \tag{7}$$

8. Technology Shock:

$$\hat{z}_t = \rho\hat{z}_{t-1} + \epsilon_t \tag{8}$$

System Reduction: Output and Labor

First, equations (2) and (6) give:

$$\hat{l}_t = \frac{1}{1+\varphi}\hat{y}_t - \frac{\sigma}{1+\varphi}\hat{c}_t$$

Combining equation (5) and the expression for $\hat{l}_t$ yields:

$$\begin{aligned} \hat{y}_t &= z_t + \alpha\hat{k}_t + (1-\alpha)\left(\frac{1}{1+\varphi}\hat{y}_t - \frac{\sigma}{1+\varphi}\hat{c}_t\right) \\ \left(1 - \frac{1-\alpha}{1+\varphi}\right)\hat{y}_t &= z_t + \alpha\hat{k}_t - \frac{(1-\alpha)\sigma}{1+\varphi}\hat{c}_t \\ \hat{y}_t &= \frac{1+\varphi}{\alpha+\varphi}z_t + \frac{1+\varphi}{\alpha+\varphi}\alpha\hat{k}_t - \frac{(1-\alpha)\sigma}{\alpha+\varphi}\hat{c}_t \end{aligned}$$

System Reduction: Capital Evolution

Taking the formula for $\hat{i}_t$ from equation (4) and substituting it into equation (3) yields:

$$\hat{k}_{t+1} = (1-\delta)k_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t - \frac{c^{ss}}{k^{ss}}\hat{c}_t$$

and using the expression for $\hat{y}_t$:

$$\hat{k}_{t+1} - (1-\delta)k_t = -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\hat{y}_t$$

$$\hat{k}_{t+1} - (1-\delta)k_t = -\frac{c^{ss}}{k^{ss}}\hat{c}_t + \frac{y^{ss}}{k^{ss}}\left(\frac{1+\varphi}{\alpha+\varphi}\hat{z}_t + \frac{1+\varphi}{\alpha+\varphi}\alpha\hat{k}_t - \frac{(1-\alpha)\sigma}{\alpha+\varphi}\hat{c}_t\right)$$

System Reduction: Euler Equation Transformation

Equation (1) can be rewritten as:

$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = \beta r^{ss} E_t\hat{r}_{t+1}$$
$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = \beta r^{ss} E_t\hat{y}_{t+1} - \beta r^{ss} E_t\hat{k}_{t+1}$$

Substituting the expression for y_t yields:

$$\sigma(E_t\hat{c}_{t+1} - \hat{c}_t) = -\beta r^{ss} E_t\hat{k}_{t+1} + \beta r^{ss}\frac{1+\varphi}{\alpha+\varphi}E_t\hat{z}_{t+1} + \dots$$
$$\dots + \beta r^{ss}\frac{1+\varphi}{\alpha+\varphi}\alpha E_t\hat{k}_{t+1} - \beta r^{ss}\frac{(1-\alpha)\sigma}{\alpha+\varphi}E_t\hat{c}_{t+1}$$

or:

$$-\sigma\hat{c}_t + \sigma\left(1 + \beta r^{ss}\frac{(1-\alpha)}{\alpha+\varphi}\right)E_t\hat{c}_{t+1} = \beta r^{ss}\left(\frac{1+\varphi}{\alpha+\varphi}\alpha - 1\right)E_t\hat{k}_{t+1} + \beta r^{ss}\frac{1+\varphi}{\alpha+\varphi}E_t\hat{z}_t \quad (9)$$

Reduced Canonical Form

We transform the original set of equations into a system of 3 equations with 3 endogenous variables (c, k, z) and 1 exogenous variable (ϵ):

$$\kappa_{11}\hat{c}_t + \kappa_{12}E_t\hat{k}_{t+1} + \kappa_{13}E_t\hat{z}_{t+1} + \kappa_{14}E_t\hat{c}_{t+1} = 0 \tag{10}$$

$$\kappa_{21}\hat{c}_t + \kappa_{22}\hat{k}_t + \kappa_{23}\hat{z}_t + \kappa_{24}\hat{k}_{t+1} = 0 \tag{11}$$

$$\hat{z}_{t+1} - \rho\hat{z}_t = \epsilon_{t+1} \tag{12}$$

where κ_{ij} contain the deep parameters. Setting $x_t = \{k_t, c_t\}$ and $v_t = z_t$ yields the form consistent with equation (A).

Blanchard and Kahn (1980)

The method solves models written as:

$$\begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + Ef_t, \tag{13}$$

where:

- x_{1t}: endogenous predetermined variables ($E_t x_{1t+1} = x_{1t+1}$).
- x_{2t}: endogenous non-predetermined variables.
- f_t: exogenous forcing variables.

Jordan Decomposition and Stability

Start with a Jordan decomposition of $\tilde{A}$:

$$\tilde{A} = \Lambda^{-1} J \Lambda, \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \tag{14}$$

where J_1 is stable ($|\lambda| < 1$) and J_2 is explosive ($|\lambda| > 1$). Partition Λ and E:

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{15}$$

Blanchard-Khan Condition: A unique solution exists if the number of explosive eigenvalues equals the number of non-predetermined variables.

Decoupling the System

Substituting the decomposition into equation (17) and pre-multiplying by Λ yields:

$$\begin{bmatrix} \bar{x}_{1t+1} \\ E_t \bar{x}_{2t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} f_t \tag{16}$$

where $\bar{x}_t = \Lambda x_t$ and $D = \Lambda E$:

$$\begin{bmatrix} \bar{x}_{1t} \\ \bar{x}_{2t} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{18}$$

Forward Iteration

For non-predetermined variables, iterate the lower portion of (26) forward:

$$\bar{x}_{2t} = J_2^{-1} E_t \bar{x}_{2t+1} - J_2^{-1} D_2 f_{2t} \tag{19}$$

Substituting recursively and using $\lim_{n\to\infty} J_2^{-n} = 0$:

$$\bar{x}_{2t} = -\sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i} \tag{20}$$

Mapping back to original variables x_{2t}:

$$x_{2t} = -\Lambda_{22}^{-1}\Lambda_{21} x_{1t} - \Lambda_{22} \sum_{i=0}^{\infty} J_2^{-(i+1)} D_2 E_t f_{2t+i} \tag{21}$$

Final BK Solution

In the case of our model ($E_t(f_{2t+i}) = \rho^i \hat{z}_t$):

$$x_{2t} = -\Lambda_{22}^{-1}\Lambda_{21}x_{1t} - \Lambda_{22}J_2^{-1}(I - \rho J_2^{-1})^{-1}D_2\hat{z}_t \tag{22}$$

Finally, the predetermined variables evolve according to:

$$x_{1t+1} = \tilde{A}_{11}x_{1t} + \tilde{A}_{12}x_{2t} + E_1 f_t \tag{23}$$

where $\tilde{A} = \Lambda^{-1}J\Lambda$.

Sims's (2001) Generalization

Method for models expressed as:

$$Ax_t = Bx_{t-1} + Cv_t + D\eta_t + E \tag{24}$$

where:

- η_t: expectational errors, $x_t = E_{t-1}x_t + \eta_t$.
- Exogenous shocks are part of x_t.
- Handles cases where A is non-invertible.

Step 1: QZ Factorization

Decompose A and B using generalized Schur decomposition:

$$A = Q' \Lambda Z' \tag{25}$$

$$B = Q' \Omega Z' \tag{26}$$

where (Q, Z) are unitary and (Λ, Ω) are upper triangular. Generalized eigenvalues are $\vartheta_i = \lambda_{ii} / \omega_{ii}$. Pre-multiplying the system by Q yields:

$$\Lambda z_t = \Omega z_{t-1} + QCv_t + QD\eta_t + QE \tag{27}$$

where $z_t = Z' x_t$.

Step 2: Partitioning the System

Partition equation (39) into explosive and non-explosive blocks:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} w_t \quad (28)$$

where $w_t = [Cv_t + D\eta_t + E]$. Generalized eigenvalues of the lower block are all outside the unit circle.

Step 3: Solving the Explosive Block

The lower block can be written as:

$$\Lambda_{22} z_{2t} = \Omega_{22} z_{2t-1} + w_{2t} \tag{29}$$

where $w_{2t} = Q_2 w_t$. Iterate forward:

$$z_{2t} = M_{22} z_{2t+1} - \Omega_{22}^{-1} w_{2t+1}, \quad M = \Omega_{22}^{-1} \Lambda_{22} \tag{30}$$

$$z_{2t} = -\sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} w_{2t+1+i} \tag{31}$$

Since $E_t(\eta_{t+s}) = E_t(v_{t+s}) = 0$ for $s > 0$:

$$z_{2t} = -\sum_{i=0}^{\infty} M^i \Omega_{22}^{-1} Q_2 E_2 \tag{32}$$

Step 3: Explosive Block Solution

Using $\sum M^i = (I - M)^{-1}$, the solution for z_{2t} is:

$$z_{2t} = (\Lambda_{22} - \Omega_{22})^{-1} Q_2 E_2 \tag{33}$$

Step 4: Uniqueness and Decoupling

Uniqueness requires a matrix Φ satisfying:

$$Q_1 D = \Phi Q_2 D \tag{34}$$

Pre-multiplying (39) by $[I \quad -\Phi]$ ensures the factor for η_t is zero:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} - \Phi\Lambda_{22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi\Omega_{22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + (Q_1 - \Phi Q_2) w_t \tag{35}$$

Step 4: Final Representation

The system becomes: $x_t = \Theta_e + \Theta_0 x_{t-1} + \Theta_1 v_t$, where:

$$H = Z \begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12} - \Phi\Lambda_{22}) \\ 0 & I \end{bmatrix} \tag{36}$$

$$\Theta_e = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ (\Omega_{22} - \Lambda_{22})^{-1} Q_2 \end{bmatrix} E \tag{37}$$

Step 4: VAR Matrices

$$\Theta_0 = Z\Lambda_{11}^{-1}[\Omega_{11} \quad (\Omega_{12} - \Phi\Omega_{22})]Z' \tag{38}$$

$$\Theta_1 = H \begin{bmatrix} Q_1 - \Phi Q_2 \\ 0 \end{bmatrix} D \tag{39}$$

This completes the procedure for expressing the DSGE model as a standard Vector Autoregression.