Lecture: Non-stationary Time Series

222061-1617: Time Series Econometrics

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Outline

Outline:

1. Unit Root
2. Structural Break
3. Unobserved Component Model and Trend/Cycle Decomposition

Unit Root

Random Walk - Example

Basic random walk

$$y_t = y_{t-1} + \varepsilon_t; \qquad \varepsilon_t \sim WN$$

Note the property

$$E_t y_{t+1} = y_t.$$

- random walks are popular in finance...
  - models for asset prices, exchange rates
- ...as well as in macroeconomics
  - permanent income hypothesis
  - unit root in macro variable: Nelson and Plosser (1982)

Random Walk: Properties

Properties of random walks:

1. The impulse-response function of a random walk is one at all horizons.

$$y_t = y_{t-1} + \varepsilon_t = y_0 + \sum_{i=1}^{t} \varepsilon_i$$

   - The impulse-response function of stationary processes dies out eventually.

2. The forecast variance of the random walk grows linearly with the forecast horizon

$$var(y_{t+k}|y_t) = var(y_{t+k} - y_t) = k\sigma^2$$

   - forecast error variance of a stationary series approaches a constant, the unconditional variance of that series.
   - the variance of the random walk is infinite.

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Random Walk: Properties

Properties of random walks:

3. The autocovariances of a random walk, seen as the limit of an AR(1), $y_t = \phi y_{t-1} + \varepsilon_t$, as $\phi \to 1$.

$$\rho_j = 1 \qquad \text{for all } j$$

   - All estimated autocorrelations are near 1; they die out “too slowly”.

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Statistical issues: Distribution of AR(1) estimates

Recall

$$y_t = y_{t-1} + \varepsilon_t; \qquad \varepsilon_t \sim WN$$

Dickey and Fuller:

- Test for a random walk performed by running $y_t = \phi y_{t-1} + \varepsilon_t$ and testing whether $\phi = 1$ not correct:
  1. OLS estimates are biased down (towards stationarity)
  2. OLS standard errors are tighter than the actual standard errors
- Many series thought to be stationary based on OLS regressions could be in fact generated by random walks.

Statistical issues: Inappropriate detrending

- Suppose the real model is

$$y_t = c + y_{t-1} + \varepsilon_t; \qquad \varepsilon_t \sim WN$$

- Suppose you include linear trend and estimate an AR(1) coefficient, i.e., fit the model

$$y_t = \alpha + \beta \cdot t + \phi y_{t-1} + \varepsilon_t$$

- OLS estimate $\hat{\phi}$ even more biased downward and standard errors more misleading.

Why?

- In a relatively small sample, the random walk is likely to drift up or down;
- Drift could well be (falsely) modeled by a linear (or nonlinear, "breaking" , etc.) trend.

Statistical issues: Spurious regression

Suppose two series are generated by independent random walks,

$$\begin{aligned} x_t &= x_{t-1} + \epsilon_t \\ y_t &= y_{t-1} + \nu_t \\ & E(\epsilon_t \nu_s) = 0 \quad \text{for all } \; t, s \end{aligned}$$

Suppose we run y_t on x_t by OLS,

$$y_t = \alpha + \beta x_t + \upsilon_t$$

- Assumptions behind the usual distribution theory are violated.
- We find statistically significant β more often than the we should.

Autoregressive Unit Root Tests

- ARMA(p,q) process:

$$\phi(L)y_t = \theta(L)\varepsilon_t, \qquad \varepsilon_t \sim WN$$

- Consider $\phi(z) = 0$, where $\phi(z)$ is a characteristic equation.

$$\begin{aligned} \phi(L) &= 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p \\ \phi(z) &= 0 \Rightarrow 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0 \\ &\Rightarrow (1 - \frac{1}{\lambda_1} z)(1 - \frac{1}{\lambda_2} z) \cdots (1 - \frac{1}{\lambda_p} z) = 0 \end{aligned}$$

- $H_0 : \phi(z) = 0$ has (at least) one root on unit circle.

Unit Root in ARMA

- If only <u>one</u> of the roots is equal to one, it can be factored out

$$\phi(z) = (1 - z)\phi^*(z),$$

$\Rightarrow \phi^*(z) = 0$ has roots outside unit circle

$$\begin{aligned} \Rightarrow \phi^*(L)(1 - L)y_t &= \phi(L)\varepsilon_t \\ \Delta y_t &= \phi^*(L)^{-1}\theta(L)\varepsilon_t \\ \Delta y_t &= \Psi^*(L)\varepsilon_t \\ \Delta y_t &= u_t, \qquad u_t = \Psi^*(L)\varepsilon_t \sim I(0) \end{aligned}$$

- n in $I(n)$ denotes order of integration.
- $I(0)$ denotes covariance stationary process.

Unit Root in ARMA

- Then

$$y_t = y_{t-1} + u_t,$$

- and, given y_0,

$$y_t = y_0 + \sum_{j=0}^{t-1} u_{t-j} \sim I(1).$$

Shocks do not die out.

- An alternative for H_0 is

$$\begin{aligned} \mathrm{H}_1 : \quad & \phi(z) = 0 \text{ has all roots outside unit circle} \\ & y_t = \phi(L)^{-1}\theta(L)\varepsilon_t \\ & y_t = \Psi(L)\varepsilon_t = u_t \sim I(0) \end{aligned}$$

Shocks will die out over time.

Brownian Motion

A Wiener process (Brownian motion) $W(\cdot)$ is a continuous-time stochastic process, associating each date $r \in [0,1]$ a scalar random variable $W(r)$ that satisfies:

1. $W(0) = 0$
2. For any dates $0 \leq t_1 \leq \ldots \leq t_k \leq 1$, the changes $W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_k) - W(t_{k-1})$ are independent normal with
$$W(s) - W(t) \sim N(0, (s-t))$$
3. $W(s)$ is continuous in s.

Intuition: A Wiener process is the scaled continuous time limit of a random walk.

Properties:

- $W(r) \sim N(0, r)$
- $\sigma W(r) \sim N(0, \sigma^2 r)$
- $W(r)^2 \sim r\chi^2(1)$

Dickey-Fuller: Case 1

Consider AR(1):

$$\begin{aligned} y_t &= \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim WN \\ & \phi(L) = 1 - \phi L \end{aligned}$$

Then

$$\begin{array}{llll} H_0: & \phi(z) = 0 & \text{has unit root} & \Leftrightarrow \quad \phi = 1 \\ H_1: & \phi(z) = 0 & \text{has roots outside unit circle} & \Leftrightarrow \quad |\phi| < 1 \end{array}$$

Standard test statistics:

$$\hat{t}_\phi = \frac{\hat{\phi} - \phi}{\hat{SE}(\hat{\phi})},$$

where $\hat{\phi}$ comes from OLS on $y_t = \hat{\phi} y_{t-1} + \hat{\varepsilon}_t$.

Dickey-Fuller Result

Testing for any $\phi \neq 1$

$$t_{\phi=0.9} = \frac{\hat{\phi} - 0.9}{\hat{SE}(\hat{\phi})} \sim t - distribution \rightarrow N(0,1)$$

Testing for $\phi = 1$:

$$t_{\phi=1} = \frac{\hat{\phi} - 1}{\hat{SE}(\hat{\phi})} \sim DF$$

$$DF \xrightarrow{d} \frac{\int_0^1 W(r)dW(r)}{(\int_0^1 W(r)^2 dr)^{1/2}}$$

- It is based on continuous time random walk process
- Both numerator and denominator are functions of r, W
- It is theoretical result: the distribution can be found numerically by simulation

Dickey-Fuller distribution

Dickey-Fuller distribution

- Does not have a closed form representation.
- Is not centered around 0.
- 5% critical value for 1-side test is -1.94 (-1.65 for Normal)
- 1% critical value for 1-side test is -2.57 (-2.32 for Normal)
- Note: -1.65 is the 9.45% quantile of the DF distribution.

Additionally,

- $\hat{\phi}$ is super-consistent: $\hat{\phi} \xrightarrow{p} \phi$ at rate T instead of the usual rate $T^{1/2}$, i.e.

$$\begin{aligned} \sqrt{T}(\hat{\phi}_T - \phi) &\xrightarrow{F} N(0, (1-\phi)^2), \qquad \text{for } |\phi| < 1 \\ \sqrt{T}(\hat{\phi}_T - 1) &\xrightarrow{p} 0, \qquad \text{degenerate distribution} \end{aligned}$$

but

$$T(\hat{\phi}_T - 1) \xrightarrow{F} DF, \qquad \text{non-degenerate distribution}$$

Superconsistency

- Note that $y_t = \varepsilon_1 + \ldots \varepsilon_T \sim N(0, \sigma^2 t)$

$$\hat{\phi}_T = \frac{\sum_{t=1}^{T} y_{t-1} y_t}{\sum_{t=1}^{T} y_{t-1}^2} \Longrightarrow \hat{\phi}_T - 1 = \frac{\sum_{t=1}^{T} y_{t-1} \varepsilon_t}{\sum_{t=1}^{T} y_{t-1}^2}, \qquad y_0 = 0.$$

- Since

$$y_t^2 = (y_{t-1} + \varepsilon_t)^2 \Longrightarrow y_{t-1}\varepsilon_t = \frac{1}{2}(y_t^2 - y_{t-1}^2 - \varepsilon_t^2),$$

then

$$\sum_{t=1}^{T} y_{t-1}\varepsilon_t = \frac{1}{2}(y_T^2 - y_0^2) - \frac{1}{2}\sum_{t=1}^{T} \varepsilon_t^2$$

- Therefore, dividing the numerator by $\sigma^2 T$, we get

$$\frac{1}{\sigma^2 T}\sum_{t=1}^{T} y_{t-1}\varepsilon_t = \frac{1}{2}\left(\frac{y_T}{\sigma\sqrt{T}}\right)^2 - \frac{1}{2\sigma^2}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2 \xrightarrow{F} \frac{1}{2}(X - 1)$$

where

$$X = \left(\frac{y_T}{\sigma\sqrt{T}}\right)^2 \quad \sim \quad \chi^2(1),$$

$$\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2 \quad \xrightarrow{p} \quad \sigma^2$$

Superconsistency

- Since $y_{t-1} \sim N(0, (t-1)\sigma^2)$, the mean of the denominator, $\sum_{t=1}^{T} y_{t-1}^2$,

$$E\left(\sum_{t=1}^{T} y_{t-1}^2\right) = \sigma^2 \sum_{t=1}^{T}(t-1) = \sigma^2 T(T-1).$$

- For $\sum_{t=1}^{T} y_{t-1}^2$ to have convergent distribution, it has to be divided by T^2.
- Therefore,

$$T(\hat{\phi}_T - \phi) = \frac{\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\varepsilon_t}{\frac{1}{T^2}\sum_{t=1}^{T} y_{t-1}^2}$$

to converge.

Nuisance parameter

- Assume

$$y_t = c + \phi y_{t-1} + \varepsilon_t$$

- For $H_0 : \quad \phi = 0$,

$$t_{\phi=0} = \frac{\phi - 0}{\hat{SE}(\hat{\phi})} \overset{A}{\sim} N(0, 1)$$

  Asymptotically, the distribution is always $N(0, 1)$, no matter what c and σ^2 are.

- If the test statistics does not depend asymptotically on other parameters (nuisance parameter) it is pivotal.
- Note: It may not be pivotal for small sample; for example, for $t = 100$ it may depend on c and/or σ^2.

Nuisance parameter in DF

- DF statistics, even asymptotically, depends on c: c is a nuisance parameter.
- Dickey and Fuller shows that if $\phi = 1$ nuisance parameters are important not only in small sample but also in asymptotic distribution (it's no longer pivotal testing).
- The small-sample distribution for DF converges to asymptotic distribution much faster than in normal case: even for $t = 100$ it will look very much like asymptotical distribution; if $\phi = 0.9$ it will require a lot of observations to get to normal distribution.

Dickey-Fuller: Case 2

True model:

$$y_t = y_{t-1} + \varepsilon_t$$

Test regression: AR(1) with constant

$$y_t = c + \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim iid\ WN$$

- Under $H_0 : \phi = 1$ and $c = 0$, $y_t \sim I(1)$ without drift

$$y_t = y_0 + \sum_{j=1}^{t} \varepsilon_j \sim I(1), \qquad y_0 = \mu,$$

- Alternatively, under $H_1 : |\phi| < 1$, $y_t \sim I(0)$

$$y_t = c + \phi y_{t-1} + \varepsilon_t \sim I(0), \qquad c = \mu(1 - \phi),$$

with shocks dying out over time.

Dickey-Fuller distribution

- The $t-$statistics is

$$t_{\phi=1}^{\mu} = \frac{\hat{\phi} - 1}{\hat{SE}(\hat{\phi})}$$

- from OLS regression $y_t = \hat{c} + \hat{\phi} y_{t-1} + \hat{\varepsilon}_t$,
- Dickey-Fuller shows that, under $H_0 : \ \phi = 1$, it is

$$t_{\phi=1}^{\mu} \xrightarrow{d} DF^{\mu} = \frac{\int_0^1 W^{\mu}(r) dW(r)}{(\int_0^1 W^{\mu}(r)^2 dr)^{1/2}},$$

- with

$$W^{\mu}(r) = W(r) - \int_0^1 W(r) dr$$

the "de-meaned" Wiener process, $\int_0^1 W^{\mu}(r) = 0$.

Remarks

- If $y_0 = \mu \neq 0$, it converge to DF^μ, if $\mu = 0$ then DF, but it doesn't matter what value of y_0 is.
- The asymptotic distributions of these test statistics are influenced by the presence (but not the value) of the constant in the test regression
- The inclusion of a constant pushes the distributions of $t^\mu_{\phi=1}$ to the left:
  - 5% critical value for 1-side test is -2.86 (-1.65 for Normal)
  - 1% critical value for 1-side test is -3.43 (-2.32 for Normal)
  - 1.65 is the 45.94% quantile of the DF^μ distribution!

Dickey-Fuller: Case 3

The test regression is

$$y_t = c + \beta \cdot t + \phi y_{t-1} + \varepsilon_t$$

and includes a constant and deterministic time trend to capture the deterministic trend under the alternative.

Hypothesis

- $H_0: \ \phi = 1, \beta = 0 : y_t \sim I(1)$ with drift

$$y_t = y_0 + c \cdot t + \sum_{j=1}^{t} \varepsilon_j \sim I(1) \text{ with drift,}$$

  where $y_0 + c \cdot t$ denotes deterministic component, and $\sum_{j=1}^{t} \varepsilon_j$ the random walk component.

- $H_1: \ |\phi| < 1$: $y_t \sim I(0)$ with deterministic time trend

$$y_t = c + \beta \cdot t + \phi y_{t-1} + \varepsilon_t \sim \text{Trend stationary}$$
$$y_t - \beta \cdot t \sim I(0)$$

Test statistics

- Test statistics

$$t^{\beta}_{\phi=1} = \frac{\hat{\phi} - 1}{\widehat{SE}(\hat{\phi})}$$

where $\hat{\phi}$ is from OLS regression

$$y_t = \hat{c} + \hat{\beta} \cdot t + \hat{\phi} y_{t-1} + \varepsilon_t.$$

- Both β and c are nuisance parameters.
- Under $H_0: \ \phi = 1$

$$t^{\beta}_{\phi=1} \xrightarrow{d} DF^{\beta} = \frac{\int_0^1 W^{\beta}(r) dW(r)}{(\int_0^1 W^{\beta}(r)^2 dr)^{1/2}},$$

with

$$W^{\beta}(r) = W^{\mu}(r) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(s - \frac{1}{2}\right) W(s) ds,$$

Test statistics

- De-meaned and detrended Wiener process.
- The inclusion of a constant and trend in the test regression further shifts the distribution of $t^{\beta}_{\phi=1}$ to the left.
  - 5% critical value for 1-side test is -3.41 (-1.65 for Normal)
  - 1% critical value for 1-side test is -3.96 (-2.32 for Normal)
  - 1.65 is the 77.52% quantile of the DF^{β} distribution!
- Test DF$^{\mu}$ has more power than DF$^{\beta}$ if $\beta = 0$.

Extending DF

- The previous unit root tests are valid if the time series y_t is well characterized by an AR(1) with white noise errors.
- Many economic and financial time series have a more complicated dynamic structure than is captured by a simple AR(1) model.
- Said and Dickey (1984) augment the basic autoregressive unit root test to accommodate general ARMA(p, q) models with unknown orders and their test is referred to as the augmented Dickey-Fuller (ADF) test

Hypothesis

- Basic AR(p) model

$$\begin{aligned} \phi(L)y_t &= \varepsilon_t, \qquad \varepsilon_t \sim WN \\ \phi(L) &= 1 - \phi_1 L - \ldots - \phi_p L^p \end{aligned}$$

- Hypothesis

$$\begin{aligned} H_0 : \quad & \phi(z) = 0 \text{ has one unit root} \\ & \phi(z) = (1-z)\phi^*(z), \qquad \phi^*(z) \text{ has no unit root.} \\ H_1 : \quad & \phi(z) = 0 \text{ has all roots outside unit circle.} \end{aligned}$$

Transformation

- Tranform $\phi(L)$

$$y_t = \rho y_{t-1} + \phi_1^* \Delta y_{t-1} + \phi_2^* \Delta y_{t-2} + \ldots + \phi_{p-1}^* \Delta y_{t-p-1} + \varepsilon_t,$$

where

$$\begin{aligned} \rho &= \phi_1 + \phi_2 + \ldots \phi_p \\ \phi_j^* &= -\sum_{k=j+1}^{p} \phi_k \end{aligned}$$

- It's just different representation of AR(p) process.
- Example: AR(2):

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \\ &= \phi_1 y_{t-1} + \phi_2 y_{t-1} - \phi_2 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t \\ &= (\phi_1 + \phi_2) y_{t-1} - \phi_2 \Delta y_{t-1} + \varepsilon_t \\ &= \rho y_{t-1} + \phi_1^* \Delta y_{t-1} + \varepsilon_t. \end{aligned}$$

Hypothesis restated

- The hypothesis can be simply restated as

$$\begin{aligned} H_0 : &\quad \rho = 1 \Leftrightarrow \text{ unit root} \\ H_1 : &\quad |\rho| < 1 \Leftrightarrow I(0) \end{aligned}$$

in equation

$$y_t = \rho y_{t-1} + u_t, \qquad u_t \sim I(0)$$

with u_t containing lagged differences to capture serial correlation in u_t.

ADF

- Augmented Dickey-Fuller Test (ADF Test)

$$\hat{t}_{\rho=1} = \frac{\hat{\rho} - 1}{\widehat{SE}(\hat{\rho})}$$

  from OLS regression

$$y_t = \hat{\rho} y_{t-1} + \hat{\phi}_1^* \Delta y_{t-1} + \ldots + \hat{\phi}_{p-1}^* \Delta y_{t-p-1} + \varepsilon_t.$$

- The distribution of t-statistics is

$$\begin{aligned} \hat{t}_{\rho=1} &\xrightarrow{d} DF \\ \hat{t}_{\rho=1}^{\mu} &\xrightarrow{d} DF^{\mu} \\ \hat{t}_{\rho=1}^{\beta} &\xrightarrow{d} DF^{\beta}. \end{aligned}$$

Intuition

Re-parameterize AR(2) model

$$\begin{aligned} y_t &= \rho y_{t-1} + \phi_1^* \Delta y_{t-1} + \varepsilon_t \\ \rho &= \phi_1 + \phi_2 \\ \phi_1^* &= -\phi_2 \end{aligned}$$

- $y_{t-1} \sim I(1) \Rightarrow \hat{\rho}$ has a non-normal, asymptotic "unit root" distribution;
- $\Delta y_{t-1} \sim I(0) \Rightarrow \hat{\phi}_1^*$ has an asymptotic normal distribution

Remarks

Remarks:

- If $\phi(L)y_t = \theta(L)\varepsilon_t$, ADF works asymptotically as p grows with sample size at rate $T^{1/3}$.
- If p unknown: choose large enough p to eliminate serial correlation in u_t in $y_t = \rho y_{t-1} + u_t$.
- If p is too small then the remaining serial correlation in the errors will bias the test.
- If p is too large then the power of the test will suffer.
- Monte Carlo experiments suggest it is better to error on the side of including too many lags.
- Choose max lag (e.g. 12 for monthly data). Test last lag with $|t_{\phi^*}| > 1.645$
- Backward selection procedure.

Phillips-Perron Unit-Root Test

Model

$$\Delta y_t = \rho y_{t-1} + u_t, \qquad u_t\text{– serially correlated residuals}$$

- We do not specify how it is correlated, do not put any parametric approach.
- If $\sum \phi^*$ is close to -1, ADF has terrible size.
- Phillips-Perron addresses this issue

Phillips-Perron Unit-Root Test

- The PP tests correct for any serial correlation and heteroskedasticity in the errors u_t of the test regression.
- It directly modifies the test statistics $t_{\rho=0}$:

$$
\begin{aligned}
Z_t &= \left(\frac{\hat{\sigma}^2}{\hat{\lambda}^2}\right)^{1/2} t_{\rho=0} - \frac{1}{2}\left(\frac{\hat{\lambda}^2 - \hat{\sigma}^2}{\hat{\lambda}^2}\right)\left(\frac{T \cdot \widehat{SE}(\hat{\rho})}{\hat{\sigma}^2}\right) \\
t_{\rho=0} &= \frac{\hat{\rho}}{\widehat{SE}(\hat{\rho})}
\end{aligned}
$$

Phillips-Perron Unit-Root Test

- Terms $\hat{\sigma}^2$ and $\hat{\lambda}^2$ are consistent estimates of the variance parameters

$$\begin{aligned} \sigma^2 &= \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} E(u_t^2) \\ \lambda^2 &= \lim_{T\to\infty} \sum_{t=1}^{T} E(T^{-1} S_T^2) = \text{“long run variance”} \\ S_T &= \sum_{t=1}^{T} u_t. \end{aligned}$$

- Result: Under the null hypothesis that $\rho = 0$, the PP Z_t statistic has the same asymptotic distributions as the ADF t-statistic.

Phillips-Perron Unit-Root Test

- The sample variance of the least squares residual $\hat{u}_t$ is a consistent estimate of σ^2.
- The Newey-West long-run variance estimate of u_t using $\hat{u}_t$ is a consistent estimate of λ^2.

$$\hat{\lambda}^2 = \hat{\gamma}_0 + 2\sum_{j=1}^{m}\left[1 - \frac{j}{m+1}\right]\hat{\gamma}_j^*$$

$$\hat{\gamma}_0 = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2$$

$$\hat{\gamma}_j^* = \frac{1}{T}\sum_{t=j+11}^{T}\hat{u}_t\hat{u}_{t-j}$$

Stationarity Tests

- Nelson-Plosser found that most macro variables have unit roots.
- They failed to reject H_0 of the presence of unit root (we may reject it because of the low power of the test).
- H_0 in unit root test is that there is unit root and N-P fails to reject it.
- We want to reverse the problem and test if the series is stationary so that unit root would make that we reject stationary H_0.
- Kwiatkowski, Phillips, Schmidt and Shin (KPSS), 1992 JoE - non parametric approach;
- Leybourne and McCabe (1994, JEBS) - parametric approach.

Unobserved Components Model

Unobserved Components Model

$$y_t = \mu_t + \varepsilon_t,$$

where

$$\begin{aligned} \mu_t &= \mu_{t-1} + u_t & u_t \sim iid(0, \sigma_u^2), \quad \mu_0 = constant, \\ \varepsilon_t &\sim I(0) & (i.e.\ \ \phi(L)\varepsilon_t = \theta(L)\eta_t) \end{aligned}$$

μ_t is unobserved component.

- it takes both cases of unit root and stationarity
- $\mu_t =$ local mean + unobserved component: it changes every time.
- Even though we don't observe this shock we can still recover it.

Stationarity Tests (KPSS)

Hypotheses:

$$\begin{aligned} H_0 : \quad & \sigma_u^2 = 0 \implies y_t \sim I(0) \\ H_1 : \quad & \sigma_u^2 > 0 \implies y_t \sim I(1) \end{aligned}$$

- Test is equivalent to testing for unit MA root in Δy_t.
- How to estimate variance $\sigma_u^2 = 0$, in model that is already difficult to estimate?

Unit MA root

- So far we talked about AR unit root: $\sigma_u^2 > 0$.
- There is alternative approach MA unit root.

Unit MA root:

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t \qquad \text{apply } (1-L) \\
(1-L)y_t &= (1-L)\mu_t + (1-L)\varepsilon_t \\
\Delta y_t &= u_t + \varepsilon_t - \varepsilon_{t-1}
\end{aligned}
$$

- We never observe the shock (unless $u_t = 0$).
- Under $H_0 : \ \sigma_u^2 = 0$, $u_t = 0$ so if there is a shock today then tomorrow will exactly offset it => no accumulation of permanent shocks.
- So unit MA root implies no permanent effect of shock.

Granger representation

If $\varepsilon_t \sim iid$, Granger representation theorem implies that

$$\Delta y_t = e_t + \theta e_{t-1},$$

where e_t is unobservable forecast error.
If $cov(u_t, \varepsilon_t) = 0$

$$cov(\Delta y_t, \Delta y_{t-1}) = cov(u_t + \varepsilon_t - \varepsilon_{t-1}, u_{t-1} + \varepsilon_{t-1} - \varepsilon_{t-2}) = -\sigma_\varepsilon^2$$

and

$$cov(\Delta y_t, \Delta y_{t-j}) = 0, \qquad \forall j > 1$$

- The same autocovariance structure as in MA(1) process.

Autocovariances

Compute autocovariances for both representation

- For

$$\Delta y_t = u_t + \varepsilon_t - \varepsilon_{t-1}$$

the autocovariances are

$$\begin{aligned} \gamma_0 &= var(\Delta y_t) = var(u_t + \varepsilon_t - \varepsilon_t) = \sigma_u^2 + 2\sigma_\varepsilon^2 \\ \gamma_1 &= cov(\Delta y_t, \Delta y_{t-1}) = -\sigma_\varepsilon^2 \\ \gamma_j &= 0, \qquad j > 1. \end{aligned}$$

- For

$$\Delta y_t = e_t + \theta e_{t-1}$$

the autocovariances are

$$\begin{aligned} \gamma_0^* &= (1+\theta^2)\sigma_e^2 \\ \gamma_1^* &= \theta\sigma_e^2 \\ \gamma_j^* &= 0, \qquad j > 1. \end{aligned}$$

Autocorrelations

- Determine the mapping from UC-ARIMA parameters to reduced form ARMA(0,1) model
  - Define $q = \frac{\sigma_u^2}{\sigma_\varepsilon^2}$ — signal-to-noise ratio
  - 1^{st}-order autocorrelations:

$$\begin{aligned} \rho_1 &= \frac{\gamma_1}{\gamma_0} = \frac{-1}{q+2}, \\ \rho_1^* &= \frac{\theta}{1+\theta^2} \end{aligned}$$

  - Set $\rho_1 = \rho_1^*$ and solve for θ

$$\theta = \frac{-(q+2) \pm \sqrt{(q+2)^2 - 4}}{2}.$$

    - Note two values of θ consistent with original model.
    - Choose the invertible solution, $|\theta| < 1$.

MA(1) representation

$$\Delta y_t = e_t + \theta e_{t-1},$$

- For $\sigma_{\varepsilon u} = 0$

$$\theta = \frac{-(q+2) + \sqrt{q^2 + 4q}}{2}, \qquad q = \frac{\sigma_u^2}{\sigma_\varepsilon^2}.$$

  q is the signal-to-noise ration.
- As $q \to 0$ we have unit MA root $\to$ we have permanent shock but they are very very small.
- If $\sigma_u^2 = 0$ then

$$q = 0 \Longrightarrow \theta = \frac{-2}{2} = -1$$

  so $\Psi^*(L) = 1 + \theta L$ has unit root.

KPSS Test

Testing

- Regress Δy_t on MA(1) process and see if $\theta = -1$.
- KPSS proposes one-sided LM statistics for hypothes

$$\begin{aligned} H_0: &\quad \sigma_u^2 = 0 \qquad \text{no random walk component, just constant} \\ H_1: &\quad \sigma_u^2 > 0 \end{aligned}$$

- LM statistics depends on process for y_t

KPSS: Case 1

- Case 1: constant term only

$$
\begin{aligned}
y_t &= \mu_t + \varepsilon_t \\
\mu_t &= \mu_{t-1} + u_t, \qquad \mu_0 = constant
\end{aligned}
$$

- Test regression

$$y_t = \alpha + \varepsilon_t \Longrightarrow \hat{\varepsilon}_t = y_t - \bar{y}$$

- LM test:

$$\hat{\eta}_\mu = \frac{1}{T^2} \sum_{t=1}^{T} \frac{S_t^2}{\Lambda^2},$$

where

  - $S_t = \sum_{j=1}^{t} \hat{\varepsilon}_j$ is a partial sum over time of residuals, and
  - Λ^2, spectral density at frequency 0

- Sum up sample residual over time $\longrightarrow$ under H_0 they should not be a big number, they should cancel out. Otherwise, (under alternative) they should get larger and larger.

KPSS: Case 1

- Under H_0 : $\sigma_u^2 = 0$

$$\eta_\mu \xrightarrow{d} \int_0^1 V(r)^2 dr$$

where

$$V(r) = W(r) - rW(1) \quad = \text{ standard Brownian Bridge}$$

- Reject at 5% if $\hat{\eta}_\mu > 0.463$.
- To do it need to estimate Λ^2, spectral density at frequency 0.
- KPSS proposes against Bartlet kernel approach:
  - depending how you choose the bandwidth you get different statistics depending on what bandwidth you choose -> big sample size distortions.
  - Some people suggest using parametric approach to construct test statistics.

KPSS: Case 2

- Case 2: constant + trend

$$\begin{aligned} y_t &= \tau \cdot t + \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim I(0) \\ \mu_t &= \mu_{t-1} + u_t, \qquad u_t \sim iid(0, \sigma_u^2) \end{aligned}$$

- Test regression

$$y_t = \alpha + \tau \cdot t + \varepsilon_t$$

- LM test:

$$\hat{\eta}_\mu = \frac{1}{T^2} \sum_{t=1}^{T} \frac{S_t^2}{\Lambda^2},$$

- Reject H_0 at 5% if $\hat{\eta}_\tau > 0.146$.

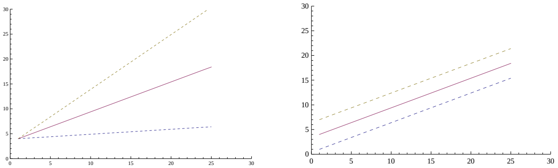
Testing

- Now we have unit root and stationarity test: apply both.
  1. Unit root test: you can't reject H_0;
     KPSS test: reject H_0.
     Both imply that series has unit root.
  2. If we can't reject both test: data give not enough observations.
  3. Reject unit root, reject stationarity: both hypothesis are component hypothesis – heteroskedasticity in series may make a big difference; if there is structural break it will affect inference.
- Power problem: if there is small random walk component (small variance σ_u^2), we can't reject unit root and can't reject stationarity.
- Economics: if the series is highly persistence we can't reject H_0 (unit root) – highly persistent may be even without unit root but it also means we shouldn't treat/take data in levels.
- If we want to quantify how important the unit root is, we should use Variance Ratio Test.

Variance ratio: Idea

- Cochrane, 1988
- non-paramteric measure of "economic" importance of unit root

Idea



- Variance of random walk grows linearly with horizon (no unconditional variance)
- Variance of trend stationary process is finite
  (it may grow over short horizon but it will finally settle down)
- Test the behavior of variance.

Variance

- Let

$$V_k = (1/k)var(Y_{t+k} - Y_t - k\mu)$$

  - V_k the variance of k^{th} period difference,
  - μ is deterministic trend that does not affect the variance.
- If there is no random walk it should converge to zero as variance is converging to constant and k is growing.
- Rewrite it as

$$\begin{aligned} V_k &= \frac{1}{k} E\left[(\Delta y_{t+1} - \mu) + (\Delta y_{t+2} - \mu) + \ldots + (\Delta y_{t+k} - \mu)\right]^2 \\ &= \gamma_0^* + 2\sum_{j=1}^{k-1} \left(\frac{k-j}{k}\right) \gamma_j^* \\ &= \text{weighted average of auto-covariances } \gamma_j^* = cov(\Delta y_t, \Delta y_{t+j}) \end{aligned}$$

- Auto-covariances are important because everything about the covariance-stationary series is in auto-covariance generating function.
- If model reduced to covariances, we can do analysis non-parametrically – just use sample auto-covariances.

Variance ratio

- Compute variance ratio

$$VR_k = \frac{V_k}{V_1}, \qquad V_1 = \gamma_0^*$$

- Result

$$\lim_{k\to\infty} V_k = \sum_{k=-\infty}^{\infty} \gamma_k^* = \Lambda^2$$

(*spectral density at frequency* 0 *for* Δy)

Example: Random Walk

- Random walk

$$y_t = y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

- Then

$$\begin{aligned} y_t &= y_0 + \sum_{j=1}^{t} \varepsilon_j, \qquad y_{t+k} = y_0 + \sum_{j=1}^{t+k} \varepsilon_j \\ y_{t+k} - y_t &= \sum_{j=t+1}^{t+k} \varepsilon_j \end{aligned}$$

- Variance

$$\begin{aligned} var(y_{t+1} - y_t) &= \sigma_\varepsilon^2, \qquad var(y_{t+k} - y_t) = k\sigma_\varepsilon^2 \\ V_1 &= \sigma_\varepsilon^2; \qquad V_k = \frac{1}{k} var(y_{t+k} - y_t) = \sigma_\varepsilon^2 \qquad \forall k \end{aligned}$$

- Variance ratio:

$$VR_k = \frac{V_k}{V_1} = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2} = 1, \qquad \forall k.$$

shock today has effect on series today and on series in the future.

Example: White Noise

- White noise

$$y_t = \varepsilon_t, \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

- Variance

$$\begin{aligned} var(y_{t+1} - y_t) &= 2\sigma_\varepsilon^2 \\ var(y_{t+k} - y_t) &= 2\sigma_\varepsilon^2 \\ V_k &= \frac{2\sigma_\varepsilon^2}{k} \qquad \forall k \end{aligned}$$

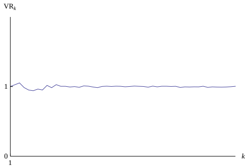
- Variance ratio

$$VR_k = \frac{1}{k} \to 0 \qquad \text{as } k \to \infty$$

- So for different type of process the variance ratio behaves differently.
- In practice, we have to estimate VR_k.
- Cochrane estimates $\widehat{VR}_k$ using Newey-West $\hat{\Lambda}_{NW}^2, \hat{\gamma}_0^*, \hat{\gamma}_j^*$.

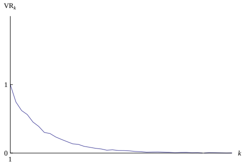
Implications

1. If $\widehat{VR}_k \to 1$



   - random walk, all shocks are permanent

2. If $\widehat{VR}_k \to 0$



   - trend stationary I(0) process

Implications

- If k too large you get spurious "mean reversion".
- In sample it always the case that

$$\hat{\gamma}_0 + \sum_{h=1}^{T-1} \hat{\gamma}_h^* = 0$$

so mean reversion has to appear.

- Which k to use? : balance the two effects.
- Non parametric approach makes problem in small sample.
- At long horizon GDP has neither $\widehat{VR} \to 0$ not $\widehat{VR} \to 1$, it's between. With standard errors, though, you can't reject any of them.

Structural Break

Structural Breaks

- Failure to reject unit root could reflect structural breaks, not unit root
- Issue: frequency of permanent shocks
  how often do you have permanent shock to the mean or drift

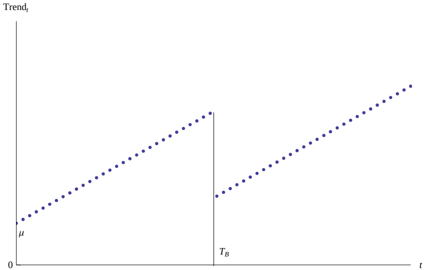
Trend stationary	$\Longrightarrow$	never
TS with one structural break	$\Longrightarrow$	once
TS with n structural breaks	$\Longrightarrow$	n times
TS with T structural breaks	$\Longrightarrow$	Unit root (difference stationary)

Perron, 1989 *Econometrica*,
"The Great Crash, The Oil Price Shock, and the Unit Root Hypothesis"

- Given a level a level break in 1929, the unit root can be rejected for 11 of 14 Nelson-Plosser (1982) series including at 10% real GNP and nominal stock prices. (annual data).
- Given the break in growth in 1973, Perron rejects unit root for postwar quarterly GDP.

Model A: The "Great Crash" Model

Model A: The "Great Crash" Model

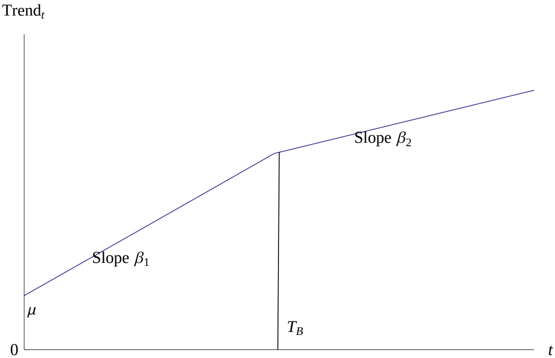


- Model of trend fluctuations of GDP.
- Trend is deterministic, when removed one gets the covariance stationary series.

$$
\begin{aligned}
Trend_t &= \mu_1 + \beta \cdot t + (\mu_2 - \mu_1) DU_t + e_t \\
DU_t &= \begin{cases} 1 & \text{if } t > T_B \\ 0 & \end{cases}
\end{aligned}
$$

Model B: The "Oil Shock" Model

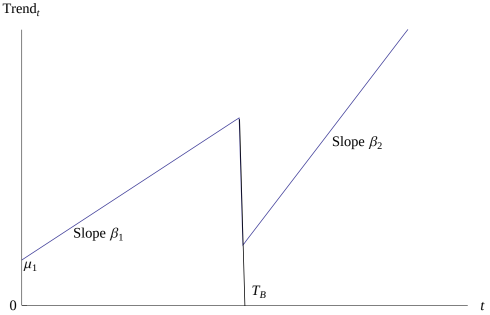
Model B: The "Oil Shock" Model



$$
\begin{aligned}
Trend_t &= \mu + \beta_1 \cdot t + (\beta_2 - \beta_1) DT_t^* + e_t \\
DT_t^* &= \begin{cases} t - T_B & \text{if } t > T_B \\ 0 \end{cases}
\end{aligned}
$$

Model C: The "Combo" Model

Model C: The "Combo" Model



$$
\begin{aligned}
Trend_t &= \mu_1 + \beta_1 \cdot t + (\mu_2 - \mu_1)DU_t + (\beta_2 - \beta_1)DT_t^* + e_t \\
DU_t &= \begin{cases} 1 & \text{if } t > T_B \\ 0 & \end{cases} \\
DT_t^* &= \begin{cases} t - T_B & \text{if } t > T_B \\ 0 & \end{cases}
\end{aligned}
$$

Test statistics

- Maybe there is more structural breaks but only one big.
- Perron: do unit root test as always, OLS:

$$y_t = \rho y_{t-1} + Trend_t(\lambda) + \sum_{k=1}^{p-1} \phi_k^* \Delta y_{t-k} + \varepsilon_t$$

  - $\lambda = \frac{T_B}{T}$ denotes location of break date.
  - Lagged difference to capture serial correlation.
- The t-statistics

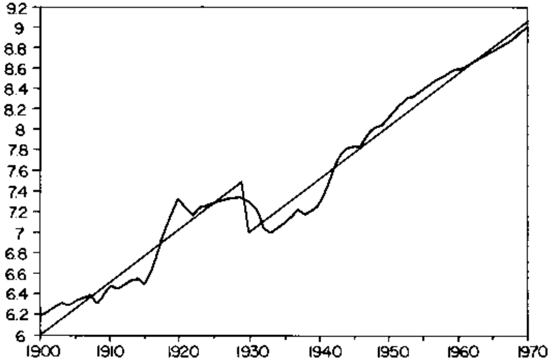
$$\hat{t}_{\rho=1}(\lambda) = \frac{\hat{\rho}(\lambda) - 1}{\widehat{SE}(\hat{\rho}(\lambda))}$$

  - We have more nuisance parameters that affect distribution

$$\hat{t}_{\rho=1}(\lambda) \overset{A}{\sim} \frac{\int_0^1 \underline{W}_\lambda(r) d\underline{W}_\lambda(r)}{\left(\int_0^1 \underline{W}_\lambda(r)^2 dr\right)^{1/2}},$$

  - $\underline{W}_\lambda$ is demeaned, detrended, dedummied Brownian motion.
  - The distribution is shifted further left than ADF.

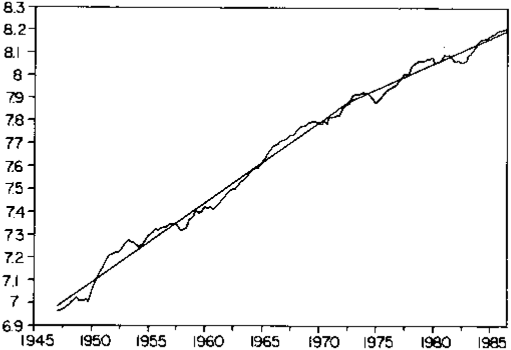
Perron (1989)



Note: The broken straight line is a fitted trend (by OLS) of the form $\tilde{y}_t = \tilde{\mu} + \tilde{\gamma} DU_t + \tilde{\beta} t$ where $DU_t = 0$ if $t \leqslant 1929$ and $DU_t = 1$ if $t > 1929$.

FIGURE 1.—Logarithm of "Nominal Wages."

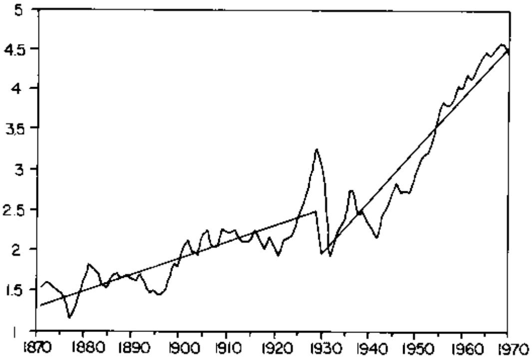
Perron (1989)



Note: The broken straight line is a fitted trend (by OLS) of the form: $\tilde{y}_t = \tilde{\mu} + \tilde{\beta} t + \tilde{\gamma} DT_t^*$ where $DT_t^* = 0$ if $t \leqslant 1973$:I and $DT_t^* = t - T_B$ if $t > 1973{:}\mathrm{I} = T_B$.

FIGURE 2.—Logarithm of "Postwar Quarterly Real GNP."

Perron (1989)



Note: The broken straight line is a fitted trend (by OLS) of the form $\tilde{y}_t = \tilde{\mu} + \tilde{\gamma}_1 DU_t + \tilde{\beta} t + \tilde{\gamma}_2 DT_t$ where $DU_t = DT_t = 0$ if $t \leqslant 1929$ and $DU_t = 1$, $DT_t = t$ if $t > 1929$.

FIGURE 3.—Logarithm of "Common Stock Prices."

Criticism

- Data mining: *How did Perron know the structural break was in 1929? He looked into data.*
- λ must be chosen independently of the data for the correct size of the test (or else there is bias against unit root, Zivot and Andrews, 1992 JBES)
  - size: if H_0 true how often you reject it
  - power: if H_1 is true (H_0 false) how often do you reject
  - *Small size and large power is optimal. We normally fix size (e.g.5% size and make test as powerful as possible.*
- $t_{5\%} = -3.8$ critical value
  λ chosen after looking at data: choosing λ so that it generates the largest t-statistics–test distributions is ever ore shifted so actual size might be bigger. Actual size might be 30% even though was set to 5%.

Criticism: Zivot and Andrews, 1992

- Need a model of break date selection procedure (Zivot and Andrews, 1992)
- $\hat{\lambda}_{INF}$ = break dates that produces the largest value of $|\hat{t}_{\rho=1}(\lambda)|$ over all λs in the sample.

$$\hat{t}_{\rho=1}(\hat{\lambda}_{INF}) = \inf_{\lambda \in \Lambda} \{\hat{t}_{\rho=1}(\lambda)\}$$

- Zivot and Andrews find that for 8 of 14 Nelson-Plosser series (including GNP) $\hat{\lambda}_{INF} = 1929$ are stationary and the t-statistics is distributed as

$$\hat{t}_{\rho=1}(\hat{\lambda}_{INF}) \overset{A}{\sim} \inf_{\lambda \in \Lambda} \left\{ \frac{\int_0^1 \underline{W}_\lambda(r) d\, \underline{W}_\lambda(r)}{\left(\int_0^1 \underline{W}_\lambda(r)^2 dr\right)^{1/2}} \right\}.$$

- It shifts distribution further left of Perron's.

Unobserved Component Model

Detrending

- Need stationary series:
$$Y_t = X_t\beta + \varepsilon_t$$
    - Granger and Newbold (1974, JoE, “Spurious Regressions in Econometrics”)
    - If y_t and X_t are independent random walk ($\beta = 0$), $\hat{\beta}_{OLS} \rightarrow$ non-zero random variable, and $\hat{t}_{\beta=0}$ is large: spurious regression phenomenon.
- Taking difference instead of levels (so we get stationary series) will bring larger standard errors => cannot reject hypothesis.
- Detrending still allows to analyze levels.
- Sometimes we are interested in trend alone.

Trend/Cycle

- Observable series y_t

$$y_t = \tau_t + c_t$$

  - τ_t is trend, and

$$\tau_t = \mu + \tau_{t-1} + \eta_t$$

  - c_t is transitory component, $(I(0))$.

- If trend contains stochastic component, random walk, then if we apply HP we get spurious cycle.
- We have two unobserved components and if we can model the cycle we can try to use unobserved component estimation.

Unobserved Components Approach

- Watson (1986, JME), Clark (1987, QJE), Morley, Nelson, Zivot (2003, ReStat)
- Approach: parametric model for c_t
- Model *("Structural")*

$$
\begin{aligned}
y_t &= \tau_t + c_t \\
\tau_t &= \mu + \tau_{t-1} + \eta_t, \qquad \eta_t \sim iidN(0, \sigma_\eta^2) \\
\phi(L)c_t &= \varepsilon_t, \qquad \varepsilon_t \sim iidN(0, \sigma_\varepsilon^2), \\
& \quad cov(\eta_t, \varepsilon_t) = \sigma_{\varepsilon\eta}
\end{aligned}
$$

Problem: Identification

- We have 1 observable series and 2 unobservable components.
- To get 2 unobservable components, we need some identification assumptions.

Identification:

- If $c_t = \varepsilon_t$ or $c_t = \phi c_{t-1} + \varepsilon_t$, then $\sigma_{\varepsilon\eta}$ is not identified *from the data*.
- There can be infinitely many values of $\sigma_{\varepsilon\eta}$ that would produce the same autocovariance generating function for the first series.
- However, that does not mean that all values of $\sigma_{\varepsilon\eta}$ are equal.
- If it is set to zero, it imposes restriction on autocovariance generating function of 1^{st} differences.

Example: AR(1)

Example: AR(1)

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \mu + \tau_{t-1} + \eta_t \\ c_t &= \phi c_{t-1} + \varepsilon_t \end{aligned}$$

- Structural model: 5 parameters: $\mu, \sigma^2_\eta, \sigma^2_\varepsilon, \phi, \sigma_{\varepsilon\eta}$.
- How many parameters can be identified from data?

Reduced-Form

- First-difference equation

$$\begin{aligned} y_t &= \tau_t + c_t \\ (1-L)y_t &= (1-L)\tau_t + (1-L)c_t \\ \Delta y_t &= \mu + \eta_t + (1-L)(1-\phi L)^{-1}\varepsilon_t \end{aligned}$$

Example: AR(1)

- Multiply both sides by $(1-\phi L)$:

$$\begin{aligned}(1-\phi L)\Delta y_t &= (1-\phi L)\mu + (1-\phi L)\eta_t + (1-L)\varepsilon_t \\ &= c + \eta_t - \phi\eta_{t-1} + \varepsilon_t - \varepsilon_{t-1}, \qquad c = (1-\phi)\mu.\end{aligned}$$

- They are unobserved but we have a sum of two iid series

$$\eta_t + \varepsilon_t + (-\phi)\eta_{t-1} + (-1)\varepsilon_{t-1}$$

- The sum of two white noise processes = white noise: same moments as MA(1).
- So this model is observationally equivalent to

$$\Delta y_t = c + \phi\Delta y_{t-1} + e_t + \theta e_{t-1}$$

- ARMA(1,1) $\Longrightarrow$ 4 parameters: $c, \phi, \theta, \sigma_e^2$, *that's how many we can estimate.*
- We have 5 parameters but only 4 observed. So far estimates assumes one of parameters fixed.

Estimation

- Assume $\sigma_{\varepsilon\eta} = 0$ (Watson, Harvey, Clark).

  => shocks that drive transitory movements are not correlated with those that drive long-run behavior.
- With this assumption the model can be estimated:
  1. Find match (functional) of observed/estimated parameters with the ones from structural model, or
  2. Cast the model in a state space form and estimate via Kalman Filter:

State-Space Form

Observation equation

$$
\begin{aligned}
y_t &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \tau_t \\ c_t \end{bmatrix} \\
y_t &= H\beta_t
\end{aligned}
$$

State equation

$$
\begin{aligned}
\begin{bmatrix} \tau_t \\ c_t \end{bmatrix} &= \begin{bmatrix} \mu \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} \tau_{t-1} \\ c_{t-1} \end{bmatrix} + \begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix}, \\
\beta_t &= \hat{\mu} + F\beta_{t-1} + e_t, \qquad e_t \sim N(0, Q), \\
& Q = \begin{bmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\varepsilon^2 \end{bmatrix}
\end{aligned}
$$

Kalman Filter: Results

- Kalman Filter does not care about how we came up with state form.
- KF: $\tau_{t|t}$ and $c_{t|t}$, $\tau_{t|T}$, $c_{t|T}$.
- We say τ_t and c_t are uncorrelated with each other, by assumption.
- $corr(\eta_{t|t}, \varepsilon_{t|t}) = -1$ even though we assume $corr(\eta_t, \varepsilon_t) = 0$.
- In classical approach $corr(x_t, \hat{\varepsilon}_t) = 0$ by construction, even though true relationship is $corr(x_t, \varepsilon_t) \neq 0$.
- Estimates of correlation rather than sample correlation of estimates.
- Identification: If we estimate the model without assuming $\sigma_{\varepsilon\eta}$ Gauss will not converge as there is ∞ many numbers of $\sigma_{\varepsilon\eta}$ for which likelihood doesn't decrease.

Morley, Nelson and Zivot (2003)

- RW + AR(2) makes model identified.

Why?

- AR(1) cycles is not observationally different from RW.
- AR(2) has this feature that cannot be proxied by RW.

Morley, Nelson and Zivot (2003):

- $\sigma_{\varepsilon\eta}$ identified for $c_t \sim ARMA(p, q)$, with $p \geq q + 2$.

Example: AR(2)

Model:

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \mu + \tau_{t-1} + \eta_t \\ c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + \varepsilon_t \end{aligned}$$

- 6 parameters: $\mu, \phi_1, \phi_2, \sigma_\eta^2, \sigma_\varepsilon^2, \sigma_{\varepsilon\eta}$.

Pre-multiplying both sides with $(1-L)$:

$$\begin{aligned} \Delta y_t &= (1-L)\tau_t + (1-L)c_t \\ &= \mu + \eta_t + (1-L)(1-\phi_1 L - \phi_2 L^2)^{-1}\varepsilon_t \\ (1-\phi_1 L - \phi_2 L^2)\Delta y_t &= (1-\phi_1-\phi_2)\mu + \eta_t - \phi_1\eta_{t-1} - \phi_2\eta_{t-2} + \varepsilon_t - \varepsilon_{t-1} \end{aligned}$$

- The model is observationally equivalent to ARMA(2,2) model: $\Delta y_t \sim ARMA(2,2)$ with 6 parameters: $c, \phi_1, \phi_2, \theta_1, \theta_2, \sigma_e^2$.

Results

- We can map parameters of ARMA(2,2) to our structural model or estimate KF with.

$$Q = \begin{bmatrix} \sigma_\eta^2 & \sigma_{\varepsilon\eta} \\ \sigma_{\varepsilon\eta} & \sigma_\varepsilon^2 \end{bmatrix}$$

- For US real GDP, setting $\sigma_{\varepsilon\eta} = 0$ can be rejected: $\rho_{\varepsilon\eta} = -0.9$.
- τ_t is volatile
- Structural model with AR(3) has 7 structural parameters but is observationally equivalent to reduced-form version ARMA(3,3) with 8 parameters: overidentification.
- Not such a big problem; $\rho_{\varepsilon\eta} < 0$ still holds.